Usual Stochastic and Hazar Rate Orderings of Fail-Safe Systems: A Multiple-Outlier Approach for Generalized Scaled Log-Logistic Model

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Abstract

This study provides a comprehensive analysis of fail-safe systems incorporating multiple generalized scaled log-logistic outlier components. In this Regard, two structural frameworks are examined: one in which system components operate independently, and another in which dependence is introduced through an Archimedean survival copula. To establish rigorous comparisons, we explore ordering relations with respect to the usual stochastic order and the hazard rate order. The investigation leverages the majorization property of generalized scaled log-logistic parameters with multiple outliers, thereby offering deeper insight into the reliability and performance assessment of fail-safe structures under both independent and dependent settings. To complement these theoretical derivations, an extensive Monte Carlo simulation study is conducted to quantify the magnitude of the reliability improvement and rigorously validate the underlying data-generating process using a suite of modern diagnostic tools.

Keywords and Phrases: Fail-safe systems; Usual stochastic order; Hazard rate order; Majorization order, Multiple-outlier models; Archimedean survival copula; Generalized scaled log-logistic model.

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1 Introduction

Let $X_1, \ldots, X_n$ be a random sample of lifetimes, and their order statistics be denoted by $X_{1:n} \leq \cdots \leq X_{n:n}$, which represent the ordered lifetimes. Order statistics play a fundamental role in numerous areas such as reliability engineering, survival analysis, actuarial science, risk management, quality control, and nonparametric statistics. In reliability theory, a classical example is provided by $X_{n-k+1:n}$, $k = 1, \ldots, n$, which denotes the lifetime of a k-out-of-n system, where the system functions as long as at least k components remain operational. Parallel, series, and fail-safe systems are special cases of k-out-of-n systems corresponding to $k = 1$, $k = n$, and $k = n - 1$, respectively. A comprehensive discussion of the properties and applications of k-out-of-n systems can be found in Barlow and Proschan (1975).

Fail-safe systems constitute an important class of reliability structures with a wide range of practical applications. A fail-safe mechanism is designed so that, in the event of a failure, the system transitions to a state that prevents catastrophic consequences. A classical illustration is the braking system of a railway train, where the brakes are kept in an off-position by air pressure. If a brake line ruptures or a carriage becomes decoupled, the loss of air pressure automatically activates the brakes through a local air reservoir, thereby ensuring safety. The reliability characteristics of fail-safe systems have attracted considerable attention in the literature; see, for example, Păltănea (2007, 2008), Zhao and Balakrishnan (2009, 2011), Balakrishnan et al. (2015), Zhang et al. (2019, 2023), Barmalzan et al. (2022, 2023), Emami et al. (2023), Hawlader et al. (2023), and Das and Kayal (2024).

For a comprehensive treatment of order statistics and their broad range of applications, the reader is referred to David and Nagaraja (2003) and Arnold et al. (1992). Traditionally, distributional characteristics such as the mean, variance, skewness, and kurtosis have been employed to compare probability distributions. However, comparisons based solely on such scalar summaries often fail to capture the overall structure of distributions. Motivated by this limitation, Mann and Whitney (1947) introduced a distribution-based framework for comparing random variables through what are now known as stochastic orders. These orders utilize the entire distributional information and have become an essential tool in reliability theory and applied probability. Further developments and applications of stochastic orders can be found in Shaked and Shanthikumar (2007) and Belzunce et al. (2015).

Stochastic comparisons of systems with independent or dependent components have been studied extensively by many authors. Relevant contributions include those of Zhao and Balakrishnan (2012), Balakrishnan et al. (2014), Cai et al. (2017), Ghanbari et al. (2021), Hosseinzadeh et al. (2022), Yan and Niu (2023), Li and Li (2023), Shekari et al. (2023a, 2023b, 2025, 2026), Guo et al. (2024), Hazra et al. (2024), Lu (2024), Shojaee et al. (2024), Yang (2024), D'Amico et al. (2025), Wang (2025), Lv et al. (2025), Das and Balakrishnan (2026), Rahaman et al. (2026), and Wang and Peng (2026).

In reliability theory, flexible lifetime distributions are particularly valuable for modeling data exhibiting diverse shapes of hazard rate functions. To achieve such flexibility, two main approaches have been commonly adopted. The first relies on classical lifetime distributions such as the gamma, Weibull, and log-normal families. The second approach enhances flexibility by introducing additional parameters to a baseline distribution, thereby generating new families with more flexibility. Among continuous lifetime distributions, the log-logistic distribution has received considerable attention due to its analytical tractability and practical relevance. Characterized by shape and scale parameters, it has found applications in survival analysis, hydrology, economics, population growth modeling, and agricultural production studies. Its closed-form cumulative distribution function (CDF) and the ability to model both increasing and decreasing hazard rates make it particularly suitable for analyzing censored lifetime data.

Recent research has focused on constructing generalized distribution families that extend well-known models while providing greater flexibility in practice. Several generator mechanisms have been proposed, including the generalized odd log-logistic family by Cordeiro et al. (2017), Zografos–Balakrishnan odd log-logistic family by Cordeiro et al. (2015), logistic-X family by Tahir et al. (2016), the Marshall-Olkin family by Marshall and Olkin (1997), beta-G family by Eugene et al. (2002), and Kumaraswamy-G family by Cordeiro and de Castro (2011).

Gleaton and Lynch (2006) introduced the generalized scaled log-logistic (GSLL) family of lifetime distributions. For a given baseline CDF $G(x)$, the GSLL distribution, denoted here by $GSLL(\alpha, \lambda, G)$, has the CDF, survival function (SF), and probability density function (PDF), respectively, given by

$$F(x) = \frac{G^{\alpha}(\lambda x)}{G^{\alpha}(\lambda x) + \bar{G}^{\alpha}(\lambda x)}, \qquad x > 0, \tag{1}$$

$$\bar{F}(x) = \frac{\bar{G}^{\alpha}(\lambda x)}{G^{\alpha}(\lambda x) + \bar{G}^{\alpha}(\lambda x)}, \qquad x > 0, \tag{2}$$

and

$$f(x) = \frac{\alpha\lambda\big(G(\lambda x)\bar{G}(\lambda x)\big)^{\alpha-1} g(\lambda x)}{\big[G^{\alpha}(\lambda x) + \bar{G}^{\alpha}(\lambda x)\big]^{2}}, \qquad x > 0, \tag{3}$$

where $g(x)$ is the PDF, $G(x)$ is the baseline CDF, $\alpha > 0$ is a shape parameter, and $\lambda > 0$ is the scale parameter. The GSLL distribution is capable of modeling asymmetric data with heavy right tails and bathtub-shaped hazard rate functions. In survival analysis, it serves as an attractive alternative to the Weibull distribution, as it can accommodate both proportional hazard and accelerated failure time models. Further properties and applications of the GSLL distribution can be found in Gleaton and Lynch (2004, 2006) and Gleaton and Rahman (2010).

Although dependence structures among system components have been extensively studied through Archimedean copulas, a comprehensive stochastic comparison of fail-safe systems with heterogeneous GSLL components under the multiple-outlier model, in both independent and dependent settings, remains largely unexplored. This paper aims to fill this gap in the literature. The main objectives of the present study are:

(1) to derive sufficient conditions for the hazard rate ordering between two fail-safe systems with independent and heterogeneous GSLL components under the multiple-outlier model, and

(2) to establish sufficient conditions for the usual stochastic ordering of such systems when the components are dependent, with their dependence structures being characterized by two different Archimedean copulas.

In this paper, we compare two fail-safe systems with heterogeneous components drawn from a GSLL distribution under the multiple-outlier framework. The rest of the paper is organized as follows. Section 2 presents preliminary concepts and some useful auxiliary results. Section 3 investigates stochastic comparisons of fail-safe systems with independent components under the multiple-outlier model. Section 4 is devoted to the comparison of fail-safe systems with dependent components. Section 5 details a comprehensive Monte Carlo simulation study designed to empirically validate our theoretical findings. Finally, some concluding remarks are provided in Section 6.

2 Definitions and Basic Notations

We now introduce briefly some well-known concepts about stochastic orders, majorization and related orders, and copulas that are most pertinent for all the developments in subsequent sections. All random variables under consideration are assumed to be continuous and non-negative, and "increasing" is used to mean "non-decreasing" and similarly "decreasing" is used to mean "non-increasing". Further, we assume all involved expectations to exist where ever they appear. For convenience, we use $\stackrel{sgn}{=}$ to denote that both sides of an equality have the same sign. We also use $G(x)$ as the baseline distribution function and $\mathbf{1}_p$ as a p-dimensional vector with all its components being 1.

Suppose X and Y are two univariate random variables with distribution functions F and G, SFs $\bar{F}(=1-F)$ and $\bar{G}(=1-G)$, PDFs f and g, and hazard rate functions $r_X(=f/\bar{F})$ and $r_Y(=g/\bar{G})$, respectively.

Definition 1 *A random variable X is said to be smaller than Y in the*

(i) usual stochastic order, denoted by $X \leq_{st} Y$, if $\bar{F}(t) \leq \bar{G}(t)$ for all $t > 0$;

(ii) hazard rate order, denoted by $X \leq_{hr} Y$, if $\frac{\bar{G}(t)}{\bar{F}(t)}$ increases in $t > 0$. If X and Y are absolutely continuous, then $X \leq_{hr} Y$ is equivalent to $r_X(t) \geq r_Y(t)$ for all $t > 0$.

The magnitude of two random variables can be compared through the concept of the usual stochastic and hazard rate orders, which are presented in the above definitions. More details on stochastic orders can be found in Shaked and Shanthikumar (2007).

Now, let I^n be an n-dimensional Euclidean space, where $I \subseteq \mathbb{R}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in I^n$ be two real vectors with ascending elements of $\{x_{(1)}, \ldots, x_{(n)}\}$ and $\{y_{(1)}, \ldots, y_{(n)}\}$, respectively.

Definition 2 *The vector $\boldsymbol{x}$ is said to*

(i) majorize the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \stackrel{m}{\succeq} \boldsymbol{y}$, if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$;

(ii) weakly supermajorize the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \stackrel{w}{\succeq} \boldsymbol{y}$, if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n$;

(iii) weakly submajorize the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \succeq_{w} \boldsymbol{y}$, if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n$.

Definition 3 *A real-valued function $\varphi : I^n \to \mathbb{R}$ is Schur-convex (Schur-concave) on I^n if $\boldsymbol{x} \stackrel{m}{\succeq} \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq (\leq) \varphi(\boldsymbol{y})$ for all $\boldsymbol{x}$, $\boldsymbol{y} \in I^n$.*

Note that $\boldsymbol{x} \stackrel{m}{\succeq} \boldsymbol{y}$ implies $\boldsymbol{x} \succeq_{w} \boldsymbol{y}$ and $\boldsymbol{x} \stackrel{w}{\succeq} \boldsymbol{y}$. For an elaborate discussion on the theory of majorization, one may refer to the book by Marshall et al. (2011).

Definition 4 *For a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ with joint distribution function H and corresponding marginal distribution functions $F_1, \ldots, F_n$, the copula of $\boldsymbol{X}$ is a distribution function $C : [0,1]^n \to [0,1]$ satisfying*

$$H(\boldsymbol{x}) = P(X_1 \le x_1, \ldots, X_n \le x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

Similarly, a survival copula of $\boldsymbol{X}$ is a SF $\hat{C} : [0,1]^n \to [0,1]$ satisfying

$$\bar{H}(\boldsymbol{x}) = P(X_1 > x_1, \ldots, X_n > x_n) = \hat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)),$$

where $\bar{H}(\boldsymbol{x})$ is a joint SF.

Definition 5 *(Nelsen, 2006, p. 151) For continuous and decreasing function $\psi : [0,\infty) \to [0,1]$, where $\psi(0) = 1$ and $\psi(+\infty) = 0$, the function $\phi = \psi^{-1}$, assumed to be its pseudo-inverse, such that*

$$C_\psi(u_1, \ldots, u_n) = \psi(\phi(u_1) + \cdots + \phi(u_n)), \qquad u_i \in (0,1), i = 1, \ldots, n,$$

is an Archimedean survival copula, with generator ψ, if $(-1)^k \psi^{(k)}(x) \ge 0$ for $k = 1, \ldots, n-2$ and $(-1)^{n-2}\psi^{(n-2)}(x)$ is decreasing and convex.

Lemma 1 *(Li and Fang (2015)) For n-dimensional Archimedean survival copulas C_{ψ_1} and C_{ψ_2}, if $\phi_2 \circ \psi_1$ is superadditive, then for all values of $\boldsymbol{u} = (u_1, \ldots, u_n) \in [0,1]^n$, $C_{\psi_1}(\boldsymbol{u}) \le C_{\psi_2}(\boldsymbol{u})$. A function f is superadditive (subadditive) if $f(x+y) \ge (\le) f(x) + f(y)$, for every x and y in the domain of f.*

Now, let us set

$$D_+ = \{(x_1, \ldots, x_n) : x_1 \ge \cdots \ge x_n > 0\},$$

$$\varepsilon_+ = \{(x_1, \ldots, x_n) : 0 < x_1 \le \cdots \le x_n\}.$$

Lemma 2 *(i) (Kundu et al. (2016), Lemma 2.1) Let $\varphi : D_+ \to \mathbb{R}$ be a function that is continuously differentiable on the interior of D_+. Then, for $\boldsymbol{x}, \boldsymbol{y} \in D_+$, $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \ge$ (resp. $\le$)$\varphi(\boldsymbol{y})$ if and only if $\varphi^{(k)}(\boldsymbol{z})$ is decreasing (resp. increasing) in $k = 1, \ldots, n$;*

(ii) (Kunda et al. (2016), Lemma 2.2) Let $\varphi : \varepsilon_+ \to \mathbb{R}$ be a function that is continuously differentiable on the interior of ε_+. Then, for $\boldsymbol{x}, \boldsymbol{y} \in \varepsilon_+$, $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \ge$ (resp. $\le$)$\varphi(\boldsymbol{y})$ if and only if $\varphi^{(k)}(\boldsymbol{z})$ is increasing (resp., decreasing) in $k = 1, \ldots, n$,

where $\varphi^{(k)}(\boldsymbol{z}) = \frac{\partial \varphi(\boldsymbol{z})}{\partial z_k}$ denotes the partial derivative of $\varphi(\boldsymbol{z})$ with respect to its kth argument.

Lemma 3 *(Marshall et al. (2011)) Let φ be a real-valued function that is defined on $\mathcal{A} \subseteq \mathbb{R}^n$. Then:*

(i) $\boldsymbol{x} \underset{w}{\succeq} \boldsymbol{y}$ leads to $\varphi(\boldsymbol{x}) \geq$ (resp. $\leq$)$\varphi(\boldsymbol{y})$ if and only if φ is increasing (resp. decreasing) and Schur-convex (resp. Schur-concave) on $\mathcal{A}$;

(ii) $\boldsymbol{x} \overset{w}{\succeq} \boldsymbol{y}$ leads to $\varphi(\boldsymbol{x}) \geq$ (resp. $\leq$)$\varphi(\boldsymbol{y})$ if and only if φ is decreasing (resp. increasing) and Schur-convex (resp. Schur-concave) on $\mathcal{A}$.

Lemma 4 *The function $f : (0, \infty) \to (0, \infty)$ given by*

$$f(x) = \frac{xa^{x-1}}{1 + a^x}$$

is increasing with respect to x, for $a > 1$.

Proof: Taking derivative of $f(x)$ with respect to x, it readily follows that

$$\frac{\partial f(x)}{\partial x} = \frac{\left[a^{x-1}(1 + x^2 \ln a)(1 + a^x) - xa^{2x-1} \ln a\right]}{(1 + a^x)^2}$$

$$= \frac{[a^{x-1}}{(1 + a^x)^2} \left[1 + a^x + x^2 \ln a\right] \geq 0.$$

The inequality follows from the facts that $x > 0$ and $\alpha > 1$, which implies that the function f is increasing in x.

3 Hazard Rate Ordering Results with Independent Components

In this section, we compare fail-safe systems with independent multiple-outlier GSLL components. The following two theorems provide different conditions for the hazard rate ordering between fail-safe systems. Let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier GSLL model with SFs $\left(\frac{\bar{G}^{\gamma_1}(\lambda x)}{G^{\gamma_1}(\lambda x)+\bar{G}^{\gamma_1}(\lambda x)}\mathbf{1}_p, \frac{\bar{G}^{\gamma_2}(\lambda x)}{G^{\gamma_2}(\lambda x)+\bar{G}^{\gamma_2}(\lambda x)}\mathbf{1}_q\right)$, where $p + q = n$, $p,\ q \geq 1$. Let $Y_1, \ldots, Y_{n^*}$ be another set of independent random variables following the multiple-outlier GSLL model with SFs $\left(\frac{\bar{G}^{\beta_1}(\lambda x)}{G^{\beta_1}(\lambda x)+\bar{G}^{\beta_1}(\lambda x)}\mathbf{1}_{p^*}, \frac{\bar{G}^{\beta_2}(\lambda x)}{G^{\beta_2}(\lambda x)+\bar{G}^{\beta_2}(\lambda x)}\mathbf{1}_{q^*}\right)$, where $p^* + q^* = n^*$, $p^*,\ q^* \geq 1$.

Under this general setup, the SFs of $X_{2:n}$ and $Y_{2:n^*}$ are, respectively, given by

$$\begin{aligned}
\bar{F}_{X_{2:n}}(x) &= \sum_{k=1}^{n}\left[\prod_{i\neq k}^{n}\left\{\frac{\bar{G}^{\gamma_i}(\lambda x)}{G^{\gamma_i}(\lambda x)+\bar{G}^{\gamma_i}(\lambda x)}\right\}\right]-(n-1)\prod_{i=1}^{n}\left\{\frac{\bar{G}^{\gamma_i}(\lambda x)}{G^{\gamma_i}(\lambda x)+\bar{G}^{\gamma_i}(\lambda x)}\right\} \\
&= p\left[\frac{\bar{G}^{\gamma_1}(\lambda x)}{G^{\gamma_1}(\lambda x)+\bar{G}^{\gamma_1}(\lambda x)}\right]^{p-1}\left[\frac{\bar{G}^{\gamma_2}(\lambda x)}{G^{\gamma_2}(\lambda x)+\bar{G}^{\gamma_2}(\lambda x)}\right]^{q} \\
&\quad +q\left[\frac{\bar{G}^{\gamma_1}(\lambda x)}{G^{\gamma_1}(x)+\bar{G}^{\gamma_1}(\lambda x)}\right]^{p}\left[\frac{\bar{G}^{\gamma_2}(\lambda x)}{G^{\gamma_2}(\lambda x)+\bar{G}^{\gamma_2}(\lambda x)}\right]^{q-1} \\
&\quad -(n-1)\left[\frac{\bar{G}^{\gamma_1}(\lambda x)}{G^{\gamma_1}(\lambda x)+\bar{G}^{\gamma_1}(\lambda x)}\right]^{p}\left[\frac{\bar{G}^{\gamma_2}(\lambda x)}{G^{\gamma_2}(\lambda x)+\bar{G}^{\gamma_2}(\lambda x)}\right]^{q} \qquad (4)
\end{aligned}$$

and

$$\begin{aligned}
\bar{F}_{Y_{2:n^*}}(x) &= \sum_{k=1}^{n^*}\left[\prod_{i\neq k}^{n^*}\left\{\frac{\bar{G}^{\beta_i}(\lambda x)}{G^{\beta_i}(\lambda x)+\bar{G}^{\beta_i}(\lambda x)}\right\}\right]-(n^*-1)\prod_{i=1}^{n^*}\left\{\frac{\bar{G}^{\beta_i}(\lambda x)}{G^{\beta_i}(\lambda x)+\bar{G}^{\beta_i}(\lambda x)}\right\} \\
&= p^*\left[\frac{\bar{G}^{\beta_1}(\lambda x)}{G^{\beta_1}(\lambda x)+\bar{G}^{\beta_1}(\lambda x)}\right]^{p^*-1}\left[\frac{\bar{G}^{\beta_2}(\lambda x)}{G^{\beta_2}(\lambda x)+\bar{G}^{\beta_2}(\lambda x)}\right]^{q^*} \\
&\quad +q^*\left[\frac{\bar{G}^{\beta_1}(\lambda x)}{G^{\beta_1}(\lambda x)+\bar{G}^{\beta_1}(\lambda x)}\right]^{p^*}\left[\frac{\bar{G}^{\beta_2}(\lambda x)}{G^{\beta_2}(\lambda x)+\bar{G}^{\beta_2}(\lambda x)}\right]^{q^*-1} \\
&\quad -(n^*-1)\left[\frac{\bar{G}^{\beta_1}(\lambda x)}{G^{\beta_1}(\lambda x)+\bar{G}^{\beta_1}(\lambda x)}\right]^{p^*}\left[\frac{\bar{G}^{\beta_2}(\lambda x)}{G^{\beta_2}(\lambda x)+\bar{G}^{\beta_2}(\lambda x)}\right]^{q^*}. \qquad (5)
\end{aligned}$$

In the following result, under sufficient conditions, stochastic comparisons of hazard rate ordering of the second-order statistics are performed in the case when the random variables are from the GSLL multiple-outlier models.

Theorem 1 *Let $X_1,\ldots,X_n$ be independent random variables following the multiple-outlier GSLL model with SFs $\left(\frac{\bar{G}^{\alpha_1}(\lambda x)}{G^{\alpha_1}(\lambda x)+\bar{G}^{\alpha_1}(\lambda x)}\mathbf{1}_p, \frac{\bar{G}^{\alpha}(\lambda x)}{G^{\alpha}(\lambda x)+\bar{G}^{\alpha}(\lambda x)}\mathbf{1}_q\right)$, where $p+q=n$, $p,\ q\geq 1$. Let $Y_1,\ldots,Y_n$ be another set of independent random variables following the multiple-outlier GSLL model with SFs $\left(\frac{\bar{G}^{\alpha_2}(\lambda x)}{G^{\alpha_2}(\lambda x)+\bar{G}^{\alpha_2}(\lambda x)}\mathbf{1}_p, \frac{\bar{G}^{\alpha}(\lambda x)}{G^{\alpha}(\lambda x)+\bar{G}^{\alpha}(\lambda x)}\mathbf{1}_q\right)$, where $p+q=n$, $p,\ q\geq 1$. If $\alpha\geq\alpha_2\geq\alpha_1$, then $Y_{2:n}\leq_{hr} X_{2:n}$.*

Proof. Assume that $G(\lambda x)=1-e^{-t}$ and $\bar{G}(\lambda x)=e^{-t}$, and so $t=-\ln\left(\bar{G}(\lambda x)\right)$. Then, according to (4) and (5), the SFs of $X_{2:n}$ and $Y_{2:n}$ can be written as

$$\bar{F}_{X_{2:n}}(t) = p\left[\frac{1}{1+(e^t-1)^{\alpha_1}}\right]^{p-1}\left[\frac{1}{1+(e^t-1)^{\alpha}}\right]^{q} + q\left[\frac{1}{1+(e^t-1)^{\alpha_1}}\right]^{p}\left[\frac{1}{1+(e^t-1)^{\alpha}}\right]^{q-1}$$
$$-(n-1)\left[\frac{1}{1+(e^t-1)^{\alpha_1}}\right]^{p}\left[\frac{1}{1+(e^t-1)^{\alpha}}\right]^{q},$$

$$\bar{F}_{Y_{2:n}}(t) = p\left[\frac{1}{1+(e^t-1)^{\alpha_2}}\right]^{p-1}\left[\frac{1}{1+(e^t-1)^{\alpha}}\right]^{q} + q\left[\frac{1}{1+(e^t-1)^{\alpha_2}}\right]^{p}\left[\frac{1}{1+(e^t-1)^{\alpha}}\right]^{q-1}$$
$$-(n-1)\left[\frac{1}{1+(e^t-1)^{\alpha_2}}\right]^{p}\left[\frac{1}{1+(e^t-1)^{\alpha}}\right]^{q},$$

Consequently, $\frac{\bar{G}^\alpha(\lambda x)}{G^\alpha(\lambda x)+\bar{G}^\alpha(\lambda x)}$ can be written in terms of t as

$$\frac{1}{(\frac{G(\lambda x)}{\bar{G}(\lambda x)})^\alpha + 1} = \frac{1}{(e^t-1)^\alpha + 1}.$$

Therefore, the hazard rate function of $X_{2:n}$ is given by

$$\begin{aligned} r_{X_{2:n}}(t) &= \frac{d\left[-\ln(\bar{F}_{X_{2:n}}(t))\right]}{dt} \\ &= \frac{pB_1\big(1+(e^t-1)^{\alpha_1}\big) + qB_2\big(1+(e^t-1)^{\alpha}\big) - (n-1)B}{p\big(1+(e^t-1)^{\alpha_1}\big) + q\big(1+(e^t-1)^{\alpha}\big) - (n-1)}, \end{aligned} \tag{6}$$

with

$$B_1 = (p-1)\Big(\frac{\alpha_1 e^t(e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q\Big(\frac{\alpha e^t(e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big),$$

$$B_2 = p\Big(\frac{\alpha_1 e^t(e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + (q-1)\Big(\frac{\alpha e^t(e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)$$

and

$$B = p\Big(\frac{\alpha_1 e^t(e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q\Big(\frac{\alpha e^t(e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big).$$

Similarly, we have

$$\begin{aligned} r_{Y_{2:n}}(t) &= \frac{d\left[-\ln(\bar{F}_{Y_{2:n}}(t))\right]}{dt} \\ &= \frac{pC_1\big(1+(e^t-1)^{\alpha_2}\big) + C_2\big(1+(e^t-1)^{\alpha}\big) - (n-1)C}{p\big(1+(e^t-1)^{\alpha_2}\big) + q\big(1+(e^t-1)^{\alpha}\big) - (n-1)}, \end{aligned} \tag{7}$$

with

$$C_1 = (p-1)\Big(\frac{\alpha_2 e^t (e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big) + q\Big(\frac{\alpha e^t (e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big),$$

$$C_2 = p\Big(\frac{\alpha_2 e^t (e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big) + (q-1)\Big(\frac{\alpha e^t (e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)$$

and

$$C = p\Big(\frac{\alpha_2 e^t (e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big) + q\Big(\frac{\alpha e^t (e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big).$$

It suffices to prove that $r_{X_{2:n}}(t) \le r_{Y_{2:n}}(t)$ for $\alpha \ge \alpha_2 \ge \alpha_1 > 0$, which is equivalent to showing that

$$\frac{pB_1\big(1+(e^t-1)^{\alpha_1}\big) + qB_2\big(1+(e^t-1)^{\alpha}\big) - (n-1)B}{p\big(1+(e^t-1)^{\alpha_1}\big) + q\big(1+(e^t-1)^{\alpha}\big) - (n-1)}$$

$$\le \frac{pC_1\big(1+(e^t-1)^{\alpha_2}\big) + qC_2\big(1+(e^t-1)^{\alpha}\big) - (n-1)C}{p\big(1+(e^t-1)^{\alpha_2}\big) + q\big(1+(e^t-1)^{\alpha}\big) - (n-1)},$$

Define the quantities Q_1, Q_2, and Q_3 as follows:

$$\begin{aligned}
Q_1 = {} & \big[p(1+(e^t-1)^{\alpha_1}) + q(1+(e^t-1)^{\alpha}) - (n-1)\big]^{-1} \\
& \times\Bigg\{p\left[(p-1)\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q\Big(\frac{\alpha e^t (e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right](1+(e^t-1)^{\alpha_1}) \\
& +q\left[p\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + (q-1)\Big(\frac{\alpha e^t (e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right](1+(e^t-1)^{\alpha}) \\
& -(n-1)\left[p\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q\Big(\frac{\alpha e^t (e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
Q_2 = {} & \big[p(1+(e^t-1)^{\alpha_2}) + q(1+(e^t-1)^{\alpha}) - (n-1)\big]^{-1} \\
& \times\Bigg\{p\left[(p-1)\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q\Big(\frac{\alpha e^t (e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right](1+(e^t-1)^{\alpha_2}) \\
& +q\left[p\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + (q-1)\Big(\frac{\alpha e^t (e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right](1+(e^t-1)^{\alpha}) \\
& -(n-1)\left[p\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q\Big(\frac{\alpha e^t (e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right]\Bigg\}
\end{aligned}$$

and

$$Q_3 = \big[p(1+(e^t-1)^{\alpha_2}) + q(1+(e^t-1)^{\alpha}) - (n-1)\big]^{-1}$$

$$\times\Bigg\{ p\left[(p-1)\Big(\frac{\alpha_2 e^t(e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big) + q\Big(\frac{\alpha e^t(e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right](1+(e^t-1)^{\alpha_2})$$

$$+q\left[p\Big(\frac{\alpha_2 e^t(e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big) + (q-1)\Big(\frac{\alpha e^t(e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right](1+(e^t-1)^{\alpha})$$

$$-(n-1)\left[p\Big(\frac{\alpha_2 e^t(e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big) + q\Big(\frac{\alpha e^t(e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right]\Bigg\}.$$

Then, it suffices to show that $Q_1 \leq Q_3$. According to Lemma 4, due to the increasing property of $\frac{\alpha e^t(e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}$ in α, for $\alpha \geq \alpha_2 \geq \alpha_1 > 0$, we have

$$\frac{\alpha_2 e^t(e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}} \geq \frac{\alpha_1 e^t(e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}},$$

and then

$$Q_3 - Q_2 \overset{sgn}{=} p\left[\frac{\alpha_2 e^t(e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}} - \frac{\alpha_1 e^t(e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\right]$$

$$\times\big[(p-1)(1+(e^t-1)^{\alpha_2}) + q(1+(e^t-1)^{\alpha}) - (n-1)\big]$$

$$\geq\ p\left[\frac{\alpha_2 e^t(e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}} - \frac{\alpha_1 e^t(e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\right]$$

$$\times[(p-1) + q - (n-1)] \geq 0,$$

which implies that $Q_2 \leq Q_3$. Furthermore, by setting

$$a = q\left[p\Big(\frac{\alpha_1 e^t(e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + (q-1)\Big(\frac{\alpha e^t(e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right](1+(e^t-1)^{\alpha})$$

$$-(n-1)\left[p\Big(\frac{\alpha_1 e^t(e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q\Big(\frac{\alpha e^t(e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right],$$

$$b = p\left[(p-1)\Big(\frac{\alpha_1 e^t(e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q\Big(\frac{\alpha e^t(e^t-1)^{\alpha-1}}{1+(e^t-1)^{\alpha}}\Big)\right],$$

$$c = q\big(1 + (e^t - 1)^{\alpha}\big) - (n-1),$$

and $d = p$, we have

$$\begin{aligned}
Q_2 - Q_1 &= \frac{b\big(1 + (e^t-1)^{\alpha_2}\big) + a}{d\big(1 + (e^t-1)^{\alpha_2}\big) + c} - \frac{b\big(1 + (e^t-1)^{\alpha_1}\big) + a}{d\big(1 + (e^t-1)^{\alpha_1}\big) + c} \\
&= \frac{(bc - ad)\left[\big(1 + (e^t-1)^{\alpha_2}\big) - \big(1 + (e^t-1)^{\alpha_1}\big)\right]}{\left[d\big(1 + (e^t-1)^{\alpha_2}\big) + c\right]\left[d\big(1 + (e^t-1)^{\alpha_1}\big) + c\right]} \\
&\overset{sgn}{=} (bc - ad)\left[\big(1 + (e^t-1)^{\alpha_2}\big) - \big(1 + (e^t-1)^{\alpha_1}\big)\right] \\
&\geq 0, \qquad (8)
\end{aligned}$$

because if $\alpha \geq \alpha_1$, then

$$\begin{aligned}
bc - ad = pq\big(1 + (e^t-1)^{\alpha}\big)\left[\frac{\alpha e^t (e^t-1)^{\alpha-1}}{1 + (e^t-1)^{\alpha}} - \frac{\alpha_1 e^t (e^t-1)^{\alpha_1 - 1}}{1 + (e^t-1)^{\alpha_1}}\right] \\
+ p(n-1)\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1 + (e^t-1)^{\alpha_1}}\Big) \geq 0,
\end{aligned}$$

and then Eq. (8) indicates that $Q_1 \leq Q_2$. Thus, $Q_1 \leq Q_3$ which gives the required result and completes the proof of the theorem. □

The following theorem examines the impact of varying sample sizes on the hazard rate function of fail-safe systems whose components are from the multiple-outlier GSLL model.

Theorem 2 *Let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier GSLL model with SFs $\Big(\frac{\bar{G}^{\alpha_1}(\lambda x)}{G^{\alpha_1}(\lambda x) + \bar{G}^{\alpha_1}(\lambda x)}\mathbf{1}_p, \frac{\bar{G}^{\alpha_2}(\lambda x)}{G^{\alpha_2}(\lambda x) + \bar{G}^{\alpha_2}(\lambda x)}\mathbf{1}_q\Big)$, where $p + q = n$, $p,\ q \geq 1$. Let $Y_1, \ldots, Y_{n^*}$ be another set of independent random variables following the multiple-outlier GSLL model with SFs $\Big(\frac{\bar{G}^{\alpha_1}(\lambda x)}{G^{\alpha_1}(\lambda x) + \bar{G}^{\alpha_1}(\lambda x)}\mathbf{1}_{p^*}, \frac{\bar{G}^{\alpha_2}(\lambda x)}{G^{\alpha_2}(\lambda x) + \bar{G}^{\alpha_2}(\lambda x)}\mathbf{1}_{q^*}\Big)$, where $p^* + q^* = n^*$, $p^*,\ q^* \geq 1$. Furthermore, suppose $p \leq p^* \leq q \leq q^*$ and $\alpha_1 \leq \alpha_2$. Then, $(p^*, q^*) \underset{w}{\succeq} (p, q)$ implies that $Y_{2:n^*} \leq_{hr} X_{2:n}$.*

Proof. Using arguments analogous to those employed in deriving the expressions in (6) and (7), we obtain the hazard rate functions of $X_{2:n}$ and $Y_{2:n^*}$, respectively, as follows:

$$r_{X_{2:n}}(t) = \frac{pD_1\big(1 + (e^t-1)^{\alpha_1}\big) + qD_2\big(1 + (e^t-1)^{\alpha_2}\big) - (n-1)D}{p\big(1 + (e^t-1)^{\alpha_1}\big) + q\big(1 + (e^t-1)^{\alpha_2}\big) - (n-1)} \overset{def}{=} \zeta(p, q),$$

where

$$D_1 = (p-1)\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q\Big(\frac{\alpha_2 e^t (e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big),$$

$$D_2 = p\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + (q-1)\Big(\frac{\alpha_2 e^t (e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big)$$

and

$$D = p\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q\Big(\frac{\alpha_2 e^t (e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big).$$

Furthermore,

$$r_{Y_{2:n^*}}(t) = \frac{p^* E_1\big(1+(e^t-1)^{\alpha_1}\big) + q^* E_2\big(1+(e^t-1)^{\alpha_2}\big) - (n^*-1)E}{p^*\big(1+(e^t-1)^{\alpha_1}\big) + q^*\big(1+(e^t-1)^{\alpha_2}\big) - (n^*-1)} \overset{def}{=} \zeta(p^*, q^*),$$

where

$$E_1 = (p^*-1)\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q^*\Big(\frac{\alpha_2 e^t (e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big),$$

$$E_2 = p^*\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + (q^*-1)\Big(\frac{\alpha_2 e^t (e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big),$$

and

$$E = p^*\Big(\frac{\alpha_1 e^t (e^t-1)^{\alpha_1-1}}{1+(e^t-1)^{\alpha_1}}\Big) + q^*\Big(\frac{\alpha_2 e^t (e^t-1)^{\alpha_2-1}}{1+(e^t-1)^{\alpha_2}}\Big).$$

Now, to establish that $Y_{2:n^*} \le_{hr} X_{2:n}$, it is necessary to show that $r_{X_{2:n}}(t) \le r_{Y_{2:n^*}}(t)$. Define

$$s_i = \frac{\alpha_i e^t (e^t-1)^{\alpha_i-1}}{1+(e^t-1)^{\alpha_i}}, \qquad r_i = 1+(e^t-1)^{\alpha_i}, \qquad t_{ij} = r_i s_j, \quad i,j = 1,2.$$

Then,

$$\zeta(p,q) = \frac{p(p-1)t_{11} + pqt_{21} + pqt_{12} + q(q-1)t_{22} - (n-1)(ps_1+qs_2)}{pr_1 + qr_2 - (n-1)}.$$

To complete the proof, it is sufficient to show that $\zeta(p,q) \le \zeta(p^*,q^*)$ under the conditions $p < p^* \le q \le q^*$ and $(p^*,q^*) \underset{w}{\succeq} (p,q)$. Differentiating $\zeta(p,q)$ with respect to p, we get

$$\frac{\partial \zeta(p,q)}{\partial p} \overset{sgn}{=} s_1\,[pr_1 + qr_2 - (n-1)] \times [(p-1)r_1 + qr_2 - (n-1)] + (pt_{11} + qt_{22})(r_1 - 1) \ge 0.$$

Similarly, we get

$$\frac{\partial \zeta(p,q)}{\partial q} \overset{sgn}{=} s_2\,[pr_1 + qr_2 - (n-1)] \times [pr_1 + (1-q)r_2 - (n-1)] + (pt_{11} + qt_{22})(r_2 - 1) \ge 0.$$

We thus have

$$
\begin{aligned}
\frac{\partial \zeta(p,q)}{\partial q} - \frac{\partial \zeta(p,q)}{\partial p} &\overset{sgn}{=} [pt_{21} + (q-1)t_{22} - (p-1)t_{11} - qt_{12} - (n-1)(s_2 - s_1)] \\
&\quad \times [pr_1 + qr_2 - (n-1)] + (pt_{11} + qt_{22})(r_2 - r_1) \\
&\geq [pt_{21} + (q-1)t_{22} - (p-1)t_{11} - qt_{12} - (n-1)(s_2 - s_1)] \\
&\quad \times [pr_1 + qr_2 - (n-1)] + (pr_1 + qr_2)(s_1 r_2 - s_1 r_1) \\
&\geq (s_2 - s_1)\,[pr_1 + (q-1)r_2 - (n-1)] + (n-1)(t_{12} - t_{11}) \geq 0,
\end{aligned}
$$

where the last two inequalities are due to the facts that $s_2 \geq s_1 \geq 0$ and $r_2 \geq r_1 \geq 1$. Therefore, $\frac{\partial \zeta(p,q)}{\partial q} - \frac{\partial \zeta(p,q)}{\partial p} \geq 0$. The desired result now follows from Part (i) of Lemma 3. $\square$

The following numerical example illustrates the result in Theorem 2.

Example 1 *Let $n = 7$, $n^* = 10$, $\alpha_1 = 0.2$, $\alpha_2 = 0.5$, $p = 2$, $q = 5$, $p^* = 3$, and $q^* = 7$. Moreover, let $\bar{G}(\lambda x) = e^{-(\lambda x)^4}$, $x > 0$, with $\lambda = 2$. It is easy to that the assumptions $\alpha_1 \leq \alpha_2$ and $(p^*, q^*) \underset{w}{\succeq} (p, q)$ stated in Theorem 2 are satisfied. Figure 1 depicts the difference between the hazard rate functions of $X_{2:7}$ and $Y_{2:10}$. It can be clearly observed that, for all $t > 0$, $r_{X_{2:7}}(t) - r_{Y_{2:10}}(t) \leq 0$. This numerical illustration confirms the validity of the result in Theorem 2.*

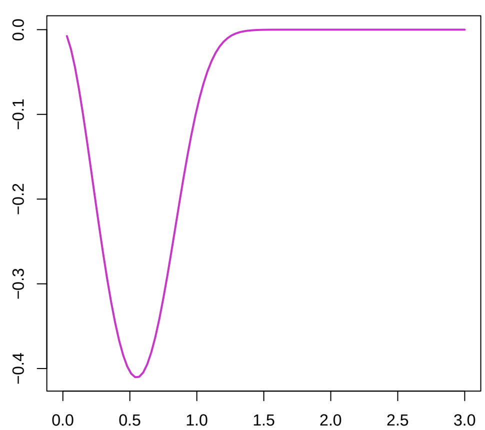

Figure 1: Graph of the difference of hazard rate functions of $X_{2:7}$ and $Y_{2:10}$.

The following remark provides an application-oriented interpretation of the above theorem.

Remark 1 *The practical implications of Theorem 2 can be illustrated through an application-oriented interpretation. Consider a communication network composed of n channels, including p high-reliability (high-cost) channels and q standard (low-cost) channels, with $p \leq q$. An enhancement policy that increases the number of high-reliability channels ($p^* \geq p$), and possibly also the number of standard channels ($q^* \geq q$), naturally satisfies the weak majorization condition $(p^*, q^*) \underset{w}{\succeq} (p, q)$. In this context, Theorem 2 provides a theoretical characterization of how such structural upgrade influences the reliability performance of the overall system. A similar interpretation arises in mechanical systems, wherein components operating under higher stress levels tend to exhibit systematically different lifetime behaviors, thereby justifying the ordering assumptions imposed on the scale parameters.*

The following counterexample illustrates that if the assumptions $p < p^* < q < q^*$ and $(p^*, q^*) \underset{w}{\succeq} (p, q)$ are not satisfied in Theorem 2, then the hazard rate ordering between two systems will not hold in general.

Counterexample 1 *Assuming that all other conditions of Example 1 remain unchanged, let $n = 8$, $n^* = 9$, $p = 3$, $q = 5$, $p^* = 2$, and $q^* = 7$. Figure 2 illustrates that, for all $t > 0$, the hazard rate function of $X_{2:8}$ is neither uniformly greater than nor uniformly smaller than that of $Y_{2:9}$. Consequently, the hazard rate ordering between these two systems does not hold in this case.*

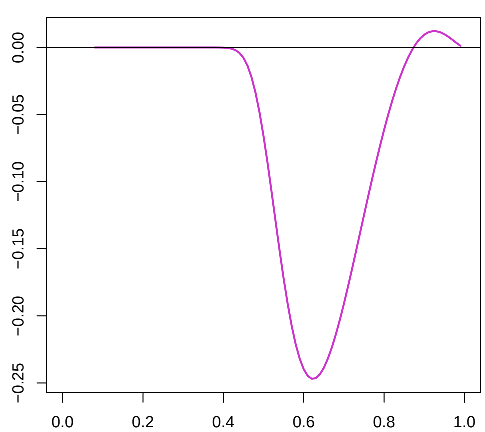

Figure 2: Graph of the difference of hazard rate functions of $r_{X_{2:8}}(t)$ and $r_{Y_{2:9}}(t)$.

4 Usual Stochastic Ordering Results with Dependent Components

In this section, two fail-safe systems are considered in the case when they both follow a multiple-outlier GSLL model. The components are supposed to be dependent, for which the dependence structure is described by an Archimedean survival copula with generator ψ. The following four theorems provide different conditions for the usual stochastic ordering to hold between the two systems.

Let $X_1, \ldots, X_n$ be dependent random variables sharing the Archimedean survival copula with generator ψ_1, following the multiple-outlier GSLL model with SFs $\left(\frac{\bar{G}_1^{\gamma_1}(\lambda_1 x)}{G_1^{\gamma_1}(\lambda_1 x)+\bar{G}_1^{\gamma_1}(\lambda_1 x)}\mathbf{1}_p, \frac{\bar{G}_1^{\gamma_2}(\lambda_2 x)}{G_1^{\gamma_2}(\lambda_2 x)+\bar{G}_1^{\gamma_2}(\lambda_2 x)}\mathbf{1}_q\right)$, where $p+q=n$, $p,\ q \geq 1$, $G_1(\lambda_i x)$ being the baseline distribution function with scale parameter λ_i, for $i = 1, 2$, and $\mathbf{1}_p$ stands for a p-dimensional vector with all its components equal to 1. Let $Y_1, \ldots, Y_{n^*}$ be another set of dependent random variables sharing the Archimedean survival copula with generator ψ_2, following the multiple-outlier GSLL model with SFs $\left(\frac{\bar{G}_2^{\beta_1}(\mu_1 x)}{G_2^{\beta_1}(\mu_1 x)+\bar{G}_2^{\beta_1}(\mu_1 x)}\mathbf{1}_{p^*}, \frac{\bar{G}_2^{\beta_2}(\mu_2 x)}{G_2^{\beta_2}(\mu_2 x)+\bar{G}_2^{\beta_2}(\mu_2 x)}\mathbf{1}_{q^*}\right)$, where $p^* + q^* = n^*$, $p^*,\ q^* \geq 1$, $G_2(\mu_i x)$ being the baseline distribution function with scale parameter μ_i, for $i = 1, 2$. Under the above general setup, the SFs of $X_{2:n}$ and $Y_{2:n^*}$ can be obtained as follows:

$$\bar{F}_{X_{2:n}}(x) = \sum_{k=1}^{n} \psi_1 \left[\sum_{i \neq k}^{n} \phi_1 \left\{\frac{\bar{G_1}^{\gamma_i}(\lambda_i x)}{G_1^{\gamma_i}(\lambda_i x) + \bar{G_1}^{\gamma_i}(\lambda_i x)}\right\}\right] - (n-1)\psi_1 \left[\sum_{i=1}^{n} \phi_1 \left\{\frac{\bar{G_1}^{\gamma_i}(\lambda_i x)}{G_1^{\gamma_i}(\lambda_i x) + \bar{G_1}^{\gamma_i}(\lambda_i x)}\right\}\right],$$

and

$$\bar{F}_{Y_{2:n^*}}(x) = \sum_{k=1}^{n^*} \psi_2 \left[\sum_{i \neq k}^{n^*} \phi_2 \left\{\frac{\bar{G_2}^{\beta_i}(\mu_i x)}{G_2^{\beta_i}(\mu_i x) + \bar{G_2}^{\beta_i}(\mu_i x)}\right\}\right] - (n^*-1)\psi_2 \left[\sum_{i=1}^{n^*} \phi_2 \left\{\frac{\bar{G_2}^{\beta_i}(\mu_i x)}{G_2^{\beta_i}(\mu_i x) + \bar{G_2}^{\beta_i}(\mu_i x)}\right\}\right].$$

In the following theorem, sufficient conditions are given for the usual stochastic order between the second-order statistics to hold. It is assumed that two observation sets have different dependence structures.

Now, we state the ordering result between the second-order statistics $X_{2:n}$ and $X_{2:n^*}$ under the condition that (p, q) is weakly submajorized by (p^*, q^*). It is assumed that the samples share an Archimedean survival copula with a common generator.

Theorem 3 *Let $X_1, \ldots, X_n$ be dependent random variables sharing the Archimedean survival copula with generator ψ_1, following the SFs $\left(\frac{\bar{G}_1^{\alpha}(\lambda_1 x)}{G_1^{\alpha}(\lambda_1 x)+\bar{G}_1^{\alpha}(\lambda_1 x)}\mathbf{1}_p, \frac{\bar{G}_2^{\alpha}(\lambda_2 x)}{G_2^{\alpha}(\lambda_2 x)+\bar{G}_2^{\alpha}(\lambda_2 x)}\mathbf{1}_q\right)$, where $p + q = n$, $p,\ q \geq 1$. Let $Y_1, \ldots, Y_{n^*}$ be dependent random variables sharing the Archimedean* *survival copula with generator ψ_1, following the SFs $\left(\frac{\bar{G}_1^{\alpha}(\lambda_1 x)}{G_1^{\alpha}(\lambda_1 x)+\bar{G}_1^{\alpha}(\lambda_1 x)}\mathbf{1}_{p^*}, \frac{\bar{G}_2^{\alpha}(\lambda_2 x)}{G_2^{\alpha}(\lambda_2 x)+\bar{G}_2^{\alpha}(\lambda_2 x)}\mathbf{1}_{q^*}\right)$, where $p^*+q^*=n^*$, p^*, $q^* \geq 1$. If $\boldsymbol{\lambda}=(\lambda_1,\lambda_2)\in D_+$, then for $G_1 \geq G_2$, $1 \leq p^* \leq p \leq q \leq q^*$ and $(p,q) \overset{m}{\succeq} (p^*,q^*)$, we have $X_{2:n} \geq_{st} Y_{2:n^*}$.*

Proof: The SFs of $X_{2:n}$ and $Y_{2:n^*}$ are given by

$$\begin{aligned}
\bar{F}_{X_{2:n}}(x) &= p\psi\left[(p-1)\phi(A_1(\lambda_1))+q\phi(A_2(\lambda_2))\right]+q\psi\left[p\phi(A_1(\lambda_1))+(q-1)\phi(A_2(\lambda_2))\right] \\
&\quad -(n-1)\psi\left[p\phi(A_1(\lambda_1))+q\phi(A_2(\lambda_2))\right] \\
&= p\psi(J_1)+q\psi(J_2)-(n-1)\psi(J_3) \\
&\overset{def}{=} \zeta_1(p,q,\psi)
\end{aligned}$$

and

$$\begin{aligned}
\bar{F}_{Y_{2:n^*}}(x) &= p^*\psi\left[(p^*-1)\phi(A_1(\lambda_1))+q^*\phi(A_2(\lambda_2))\right]+q^*\psi\left[p^*\phi(A_1(\lambda_1))+(q^*-1)\phi(A_2(\lambda_2))\right] \\
&\quad -(n^*-1)\psi\left[p^*\phi(A_1(\lambda_1))+q^*\phi(A_2(\lambda_2))\right] \\
&= p^*\psi(J_1^*)+q^*\psi(J_2^*)-(n^*-1)\psi(J_3^*) \\
&\overset{def}{=} \zeta_2(p^*,q^*,\psi),
\end{aligned}$$

where

$$A_i(\lambda_i)=\frac{\bar{G}_i^{\alpha}(\lambda_i x)}{G_i^{\alpha}(\lambda_i x)+\bar{G}_i^{\alpha}(\lambda_i x)}, \quad i=1,2,$$

$$J_1=(p-1)\phi(A_1(\lambda_1))+q\phi(A_2(\lambda_2)),$$

$$J_2=p\phi(A_1(\lambda_1))+(q-1)\phi(A_2(\lambda_2)),$$

$$J_3=p\phi(A_1(\lambda_1))+q\phi(A_2(\lambda_2)),$$

$$J_1^*=(p^*-1)\phi(A_1(\lambda_1))+q^*\phi(A_2(\lambda_2)),$$

$$J_2^* = p^*\phi(A_1(\lambda_1)) + (q^* - 1)\phi(A_2(\lambda_2))$$

and

$$J_3^* = p^*\phi(A_1(\lambda_1)) + q^*\phi(A_2(\lambda_2)).$$

To establish the desired result, it is sufficient to show that $\bar{F}_{X_{2:n}}(x) \leq \bar{F}_{Y_{2:n^*}}(x)$ and according to Part (i) of Lemma 3, it must be shown that $\zeta_1(p,q,\psi)$ is decreasing and Schur-concave in (p,q). From $\lambda_1 \geq \lambda_2$, $G_1 \geq G_2$, and the increasing property of distribution function, we conclude that, for $x > 0$, we have

$$G_1(\lambda_1 x) \geq G_1(\lambda_2 x) \geq G_2(\lambda_2 x).$$

Therefore, by the decreasing property of $A_i(\lambda_i)$ in G_i, for $i = 1, 2$, we have

$$A_1(\lambda_1 x) \leq A_1(\lambda_2 x) \leq A_2(\lambda_2 x),$$

Thus, the decreasing property of ϕ implies $\phi(A_1(\lambda_1 x)) \geq \phi(A_2(\lambda_2 x))$. On the other hand, since $J_1 = J_2 + [\phi(A_2(\lambda_2 x)) - \phi(A_1(\lambda_1 x))]$ therefore, we conclude that $J_1 \leq J_2$. By differentiating function $\bar{F}_{X_{2:n}}(x)$ with respect to p and q, we get

$$\frac{\partial\zeta_1(p,q,\psi)}{\partial p} = \psi(J_1) + p\psi'(J_1)\phi(A_1(\lambda_1)) + q\psi'(J_2)\phi(A_1(\lambda_1))1$$

$$-(n-1)\phi(A_1(\lambda_1))\psi'(J_3)$$

and

$$\frac{\partial\zeta_1(p,q,\psi)}{\partial q} = \psi(J_2) + p\psi'(J_1)\phi(A_2(\lambda_2)) + q\psi'(J_2)\phi(A_2(\lambda_2))$$

$$-(n-1)\phi(A_2(\lambda_2))\psi'(J_3).$$

Therefore, ζ_1 is Schur-concave in (p,q). Then, for $p \leq q$, we have

$$\frac{\partial\zeta_1(p,q,\psi)}{\partial p} - \frac{\partial\zeta_1(p,q,\psi)}{\partial q} = [\psi(J_1) - \psi(J_2)] + [\phi(A_1(\lambda_1)) - \phi(A_2(\lambda_2))]$$

$$\times \left[p\psi'(J_1) + q\psi'(J_2) - (n-1)\psi'(J_3)\right] \leq 0,$$

which completes the proof of the theorem. □

The next example provides a demonstration of the Theorem 3.

Example 2 *Suppose $X_1, \ldots, X_4$ are dependent random variables following the GSLL model with a common generating function $\psi(x) = e^{\frac{1-e^x}{0.95}}$. Further, let $G_1(x) = 1 - e^{-4x}$ and $G_2(x) = 1 - e^{-3x}$, $x > 0$, be the baseline distribution functions. Clearly, $G_1(x) \geq G_2(x)$. Now, set $p = 2$, $q = 4$, $p^* = 3$, $q^* = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$, and $\alpha = 4$. It is easy to observe that $(p, q) \overset{m}{\succeq} (p^*, q^*)$. Figure 3 displays the SFs of the random variables $(X_{2:6}+1)^{-1}$ and $(Y_{2:6}+1)^{-1}$. As the figure shows that the curve of $\bar{F}_{(X_{2:6}+1)^{-1}}(x)$ always lies below the curve of $\bar{F}_{(Y_{2:6}+1)^{-1}}(x)$. Thus, $(X_{2:6}+1)^{-1} \leq_{st} (Y_{2:6}+1)^{-1}$, and consequently $X_{2:6} \geq_{st} Y_{2:6}$.*

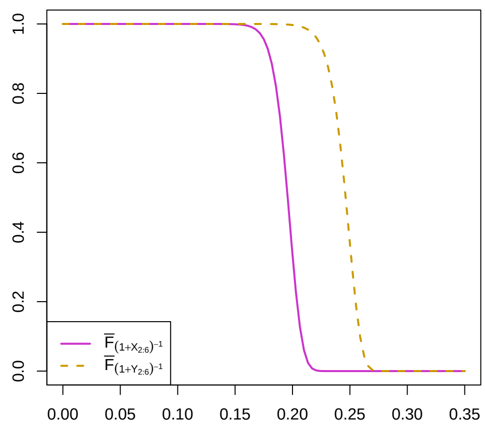

Figure 3: Plots of SFs of random variables $(X_{2:6}+1)^{-1}$ and $(Y_{2:6}+1)^{-1}$.

Theorem 4 *Let $X_1, \ldots, X_n$ be a set of non-negative dependent random variables sharing the Archimedean survival copula with generator ψ_1, following the SFs $\frac{\bar{G}^{\alpha_i}(\lambda x)}{G^{\alpha_i}(\lambda x)+\bar{G}^{\alpha_i}(\lambda x)}$, for $i = 1, \ldots, n$. Also, let $Y_1, \ldots, Y_n$ be another set of non-negative dependent random variables sharing the Archimedean survival copula with generator ψ_2, following the SFs $\frac{\bar{G}^{\beta_i}(\lambda x)}{G^{\beta_i}(\lambda x)+\bar{G}^{\beta_i}(\lambda x)}$, for $i = 1, \ldots, n$. Moreover, assume that ψ_1 or ψ_2 is log-concave, $\boldsymbol{\alpha} \in \varepsilon_+$, and $\boldsymbol{\beta} \in \varepsilon_+$. Then, we have:*

(i) $\boldsymbol{\alpha} \overset{w}{\preceq} \boldsymbol{\beta}$ *implies that $X_{2:n} \leq_{st} Y_{2:n}$ provided $\phi_2 \circ \psi_1$ is superadditive;*

(ii) $\boldsymbol{\alpha} \underset{w}{\succeq} \boldsymbol{\beta}$ *implies that $X_{2:n} \geq_{st} Y_{2:n}$ provided $\phi_2 \circ \psi_1$ is subadditive.*

Proof:

(i) The SFs of $X_{2:n}$ and $Y_{2:n}$ can be written as

$$\begin{aligned}\bar{F}_{X_{2:n}}(x) &= \sum_{i=1}^{n} \psi_1 \left[\sum_{j\neq i}^{n} \phi_1(A_{\alpha_j})\right] - (n-1)\psi_1 \left[\sum_{i=1}^{n} \phi_1(A_{\alpha_i})\right] \\ &\overset{def}{=} \Lambda_1(\boldsymbol{\alpha}, \psi_1, x)\end{aligned}$$

and

$$\begin{aligned}\bar{F}_{Y_{2:n}}(x) &= \sum_{i=1}^{n} \psi_2 \left[\sum_{j\neq i}^{n} \phi_2(A_{\beta_j})\right] - (n-1)\psi_2 \left[\sum_{i=1}^{n} \phi_2(A_{\beta_i})\right] \\ &\overset{def}{=} \Lambda_2(\boldsymbol{\beta}, \psi_2, x),\end{aligned}$$

where $A_{\alpha_i} = \frac{\bar{G}^{\alpha_i}(\lambda x)}{G^{\alpha_i}(\lambda x)+\bar{G}^{\alpha_i}(\lambda x)}$ and $A_{\beta_i} = \frac{\bar{G}^{\beta_i}(\lambda x)}{G^{\beta_i}(\lambda x)+\bar{G}^{\beta_i}(\lambda x)}$, for $i = 1, \ldots, n$. It is easy to observe that A_{α_i} and A_{β_i} are decreasing and convex with respect to α_i and β_i. To obtain the desired result, it suffices to show that $\Lambda_1(\boldsymbol{\alpha}, \psi_1, x) \leq \Lambda_2(\boldsymbol{\beta}, \psi_2, x)$. By applying the superadditive property of $\phi_2 \circ \psi_1$ and Lemma 1, we can get

$$\Lambda_1(\boldsymbol{\beta}, \psi_1, x) \leq \Lambda_2(\boldsymbol{\beta}, \psi_2, x).$$

Therefore, to obtain the required result, it suffices to show that

$$\Lambda_1(\boldsymbol{\alpha}, \psi_1, x) \leq \Lambda_1(\boldsymbol{\beta}, \psi_1, x),$$

which is equivalent to proving that Λ_1 is Schur-convex in $\boldsymbol{\alpha}$. Now, for any $k = 1, \ldots, n$, taking the partial derivative of $\Lambda_1(\boldsymbol{\alpha}, \psi_1, x)$ with respect to α_k, we have

$$\begin{aligned}\frac{\partial \Lambda_1(\boldsymbol{\alpha}, \psi_1, x)}{\partial \alpha_k} &= \frac{A'_{\alpha_k}}{A_{\alpha_k}} \times \frac{\psi_1\Big(\phi_1(A_{\alpha_k})\Big)}{\psi'_1\Big(\phi_1(A_{\alpha_k})\Big)} \times \left[\sum_{i\neq k}^{n} \psi'_1\Big(\sum_{j\neq i}^{n} \phi_1(A_{\alpha_j})\Big) - (n-1)\psi'_1\Big(\sum_{i=1}^{n} \phi_1(A_{\alpha_i})\Big)\right] \\ &\overset{def}{=} B_{\alpha_k} C_k, \end{aligned} \tag{9}$$

where $A'_{\alpha_k} = \frac{\partial A_{\alpha_k}}{\partial \alpha_k} \leq 0$, $B_{\alpha_k} = \frac{A'_{\alpha_k}}{A_{\alpha_k}} \times \frac{\psi_1\Big(\phi_1(A_{\alpha_k})\Big)}{\psi'_1\Big(\phi_1(A_{\alpha_k})\Big)} \geq 0$, and $C_k = \sum\limits_{i\neq k}^{n} \psi'_1\Big(\sum\limits_{j\neq i}^{n} \phi_1(A_{\alpha_j})\Big) - (n-1)\psi'_1\Big(\sum\limits_{i=1}^{n} \phi_1(A_{\alpha_i})\Big) \leq 0$. The expression in (9) is nonpositive due to the fact that

$A'_{\alpha_k} \leq 0$, $\psi'_1 \leq 0$, and $C_k \leq 0$; and so Λ_1 is decreasing in $\boldsymbol{\alpha}$. Hence, we have

$$\frac{\partial \Lambda_1(\boldsymbol{\alpha}, \psi_1, x)}{\partial \alpha_k} - \frac{\partial \Lambda_1(\boldsymbol{\alpha}, \psi_1, x)}{\partial \alpha_l} = B_{\alpha_k} C_k - B_{\alpha_l} C_l$$

$$= B_{\alpha_k}(C_k - C_l) + (B_{\alpha_k} - B_{\alpha_l})C_l. \tag{10}$$

According to Lemma 2, in order to establish the Schur-convexity of Λ_1 in $\boldsymbol{\alpha}$, it suffices to show that (10) is non-positive, for any $1 \leq k < l \leq n$ and $\alpha_k < \alpha_l$. For this purpose, since $C_L \leq 0$ (a proof is provided in Appendix 1), we have

$$(B_{\alpha_k} - B_{\alpha_l})C_l \stackrel{sgn}{=} B_{\alpha_l} - B_{\alpha_k}$$

$$= \frac{A'_{\alpha_l}}{A_{\alpha_l}} . \frac{\psi_1\Big(\phi_1(A_{\alpha_l})\Big)}{\psi'_1\Big(\phi_1(A_{\alpha_l})\Big)} - \frac{A'_{\alpha_l}}{A_{\alpha_k}} . \frac{\psi_1\Big(\phi_1(A_{\alpha_k})\Big)}{\psi'_1\Big(\phi_1(A_{\alpha_k})\Big)}$$

$$\leq 0. \tag{11}$$

The reason for the nonpositivity of (11) is given in Appendix 2. Similarly, we have

$$B_{\alpha_k}(C_k - C_l) \stackrel{sgn}{=} C_k - C_l$$

$$= \sum_{i \neq k}^{n} \psi'_1\Big(\sum_{j \neq i}^{n} \phi_1(A_{\alpha_j})\Big) - \sum_{i \neq l}^{n} \psi'_1\Big(\sum_{j \neq i}^{n} \phi_1(A_{\alpha_j})\Big)$$

$$= \psi'_1\Big(\sum_{j \neq l}^{n} \phi_1(A_{\alpha_j})\Big) - \psi'_1\Big(\sum_{j \neq k}^{n} \phi_1(A_{\alpha_j})\Big)$$

$$\leq 0, \tag{12}$$

which is due to the convexity of ψ_1 and the decreasing property of A_{α_k}, for $k = 1, \ldots, n$. The nonpositivity of (12) is established in Appendix 3. Therefore, $B_{\alpha_k}(C_k - C_l) + (B_{\alpha_k} - B_{\alpha_l})C_l \leq 0$. Thus, Λ_1 is Schur-convex, and then the desired result follows immediately from Part (ii) of Lemma 3.

(ii) For Part (ii) of the theorem, we note that Λ_1 is Schur-convex in $\boldsymbol{\alpha}$. Therefore, the claimed result follows from Part (i) of Lemma 3. The remaining arguments proceed analogously to those for Part (i) and are omitted for brevity.

□

The following example provides a demonstration for the two results in Theorem 4.

Example 3 *(i) Suppose* X_1, X_2, X_3, X_4 *and* Y_1, Y_2, Y_3, Y_4 *are two dependent random sets from the GSLL model with the specifications mentioned in Theorem 4. Further, consider* $\psi_1(x) = e^{\frac{1-e^x}{\theta_1}}$ *and* $\psi_2(x) = e^{\frac{1-e^x}{\theta_2}}$ *to be the generator functions, which are both log-concave. Clearly, for every* $\theta_1 \geq \theta_2 > 0$, $\phi_2 \circ \psi_1(x) = \log(1 - \frac{\theta_2}{\theta_1}(1 - e^x))$ *is a convex function, so that* $\phi_2 \circ \psi_1$ *is superadditive. Now, let* $G(\lambda x) = 1 - e^{\lambda x}$ *with* $\lambda = 2$ *and* $x > 0$ *to be the baseline CDF. Upon setting* $p = 2, q = 2, \theta_1 = 0.95, \theta_2 = 0.35, \alpha_1 = 3, \alpha_2 = 7, \beta_1 = 2,$ *and* $\beta_2 = 8$, *it can be seen that* $\boldsymbol{\beta} = (2, 2, 8, 8) \overset{w}{\succeq} (3, 3, 7, 7) = \boldsymbol{\alpha}$. *Thus, all conditions of Theorem 4 hold. Figure 4 plots the CDFs of random variables* $(1 + X_{2:4})^{-1}$ *and* $(1 + Y_{2:4})^{-1}$, *from which it can be observed that* $F_{(1+X_{2:4})^{-1}}(x)$ *is always smaller than* $F_{(1+Y_{2:4})^{-1}}(x)$, *which confirms that* $X_{2:4} \leq_{st} Y_{2:4}$.

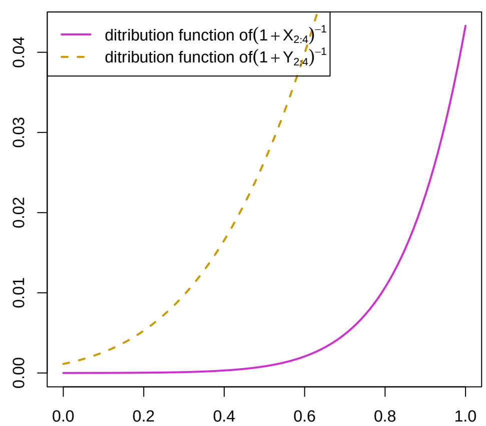

Figure 4: Plots of CDFs of random variables $(1 + X_{2:4})^{-1}$ and $(1 + Y_{2:4})^{-1}$.

(ii) Upon setting $\theta_1 = 0.35, \theta_2 = 0.95$ *in Part (i),* $\phi_2 \circ \psi_1$ *will become a subadditive function. Further, we set* $\boldsymbol{\beta} = (3, 3, 7, 7)$ *and* $\boldsymbol{\alpha} = (2, 2, 8, 8)$. *Figure 5 plots the CDFs of random variables* $(1 + X_{2:4})^{-1}$ *and* $(1 + Y_{2:4})^{-1}$, *from which it can be observed that* $F_{(1+Y_{2:4})^{-1}}(x)$ *is always smaller than* $F_{(1+X_{2:4})^{-1}}(x)$, *and this verifies that* $X_{2:4} \geq_{st} Y_{2:4}$.

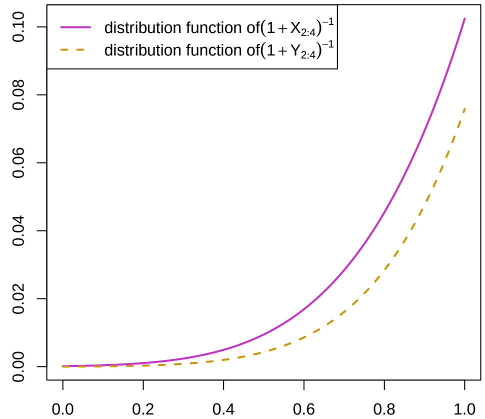

Figure 5: Plots of CDFs of random variables $(1+X_{2:4})^{-1}$ and $(1+Y_{2:4})^{-1}$.

The following counterexample shows that ignoring the subadditive assumption in Part (ii) of Theorem 4 violates the result of the theorem.

Counterexample 2 *Upon setting $\theta_2 = 0.35, \theta_1 = 0.95$ in Part (ii) of Example 3, $\phi_2 \circ \psi_1$ becomes a superadditive function. Figure 6 reveals that $\bar{F}_{X_{2:4}}(x) - \bar{F}_{Y_{2:4}}(x)$ is not always a nonnegative or nonpositive function. Thus, $Y_{2:4} \nleq_{st} X_{2:4}$ and $Y_{2:4} \ngeq_{st} X_{2:4}$.*

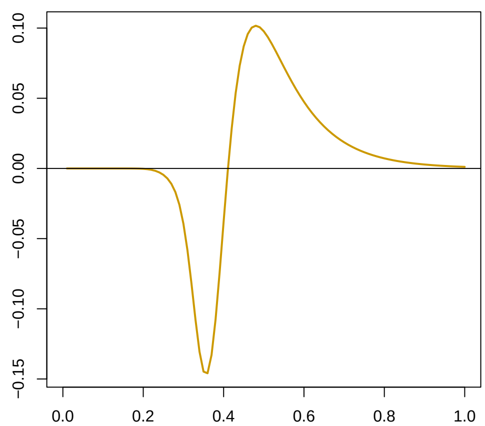

Figure 6: Plot of $\bar{F}_{X_{2:4}}(x) - \bar{F}_{Y_{2:4}}(x)$.

Theorem 5 *Let $X_1, \ldots, X_n$ be dependent random variables sharing the Archimedean survival* *copula with generator ψ_1, following the SFs $\left(\frac{\bar{G}_1^\alpha(\lambda_1 x)}{G_1^\alpha(\lambda_1 x)+\bar{G}_1^\alpha(\lambda_1 x)}\mathbf{1}_p, \frac{\bar{G}_2^\alpha(\lambda_2 x)}{G_2^\alpha(\lambda_2 x)+\bar{G}_2^\alpha(\lambda_2 x)}\mathbf{1}_q\right)$, where $p+q=n$, $p,\ q\geq 1$. Let $Y_1,\ldots,Y_{n^*}$ be dependent random variables sharing the Archimedean survival copula with generator ψ_1, following the SFs $\left(\frac{\bar{G}_1^\alpha(\mu_1 x)}{G_1^\alpha(\mu_1 x)+\bar{G}_1^\alpha(\mu_1 x)}\mathbf{1}_p, \frac{\bar{G}_2^\alpha(\mu_2 x)}{G_2^\alpha(\mu_2 x)+\bar{G}_2^\alpha(\mu_2 x)}\mathbf{1}_q\right)$. Suppose r_1 and r_2 are the hazard rate functions corresponding to G_1 and G_2, respectively. If $r_1(x)\leq r_2(x)$, $r_1(x)$ or $r_2(x)$ is decreasing, $\phi_2\circ\psi_1$ is superadditive, ψ_1 or ψ_2 is log-convex, $G_1\geq G_2$, and $\boldsymbol{\lambda}=(\lambda_1,\lambda_2),\ \boldsymbol{\mu}=(\mu_1,\mu_2)\in D_+$, then $(\mu_1,\mu_2)\overset{m}{\preceq}(\lambda_1,\lambda_2)$ implies that $X_{2:n}\leq_{st} Y_{2:n}$.*

Proof: The SFs of $X_{2:n}$ and $Y_{2:n}$ are as follows:

$$\begin{aligned}
\bar{F}_{X_{2:n}}(x) &= p\psi_1\left[(p-1)\phi_1(A_1(\lambda_1))+q\phi_1(A_2(\lambda_2))\right]+q\psi_1\left[p\phi_1(A_1(\lambda_1))+(q-1)\phi_1(A_2(\lambda_2))\right]\\
&\quad -(n-1)\psi_1\left[p\phi_1(A_1(\lambda_1))+q\phi_1(A_2(\lambda_2))\right]\\
&\overset{def}{=} \Lambda_3(\boldsymbol{\lambda},\psi_1,x)
\end{aligned}$$

and

$$\begin{aligned}
\bar{F}_{Y_{2:n}}(x) &= p\psi_2\left[(p-1)\phi_2(A_1(\mu_1))+q\phi_2(A_2(\mu_2))\right]+q\psi_2\left[p\phi_2(A_1(\mu_1))+(q-1)\phi_2(A_2(\mu_2))\right]\\
&\quad -(n-1)\psi_2\left[p\phi_2(A_1(\mu_1))+q\phi_2(A_2(\mu_2))\right]\\
&\overset{def}{=} \Lambda_4(\boldsymbol{\mu},\psi_2,x).
\end{aligned}$$

Now, it suffices to show that $\Lambda_3(\boldsymbol{\lambda},\psi_1,x)\leq\Lambda_4(\boldsymbol{\mu},\psi_2,x)$. According to Lemma 1, the superadditive property of $\phi_2\circ\psi_1$ yields

$$\Lambda_3(\boldsymbol{\mu},\psi_1,x)\leq\Lambda_4(\boldsymbol{\mu},\psi_2,x).$$

Therefore, to achieve the required result, it suffices to show that

$$\Lambda_3(\boldsymbol{\lambda},\psi_1,x)\leq\Lambda_3(\boldsymbol{\mu},\psi_1,x),$$

which is equivalent to proving that $\Lambda_3(\boldsymbol{\lambda},\psi_1,x)$ is Schur-concave in $\boldsymbol{\lambda}\in D_+$. By differentiating $\Lambda_3(\boldsymbol{\lambda},\psi_1,x)$ with respect to λ_1 and λ_2, we have

$$\begin{aligned}
\frac{\partial\Lambda_3(\boldsymbol{\lambda},\psi_1,x)}{\partial\lambda_1} &= \frac{A_1'(\lambda_1)}{A_1(\lambda_1)}\times\frac{\psi_1\left[\phi_1(A_1(\lambda_1))\right]}{\psi_1'\left[\phi_1(A_1(\lambda_1))\right]}\times\left[p(p-1)\psi_1'(I_1)+pq\psi_1'(I_2)-(n-1)p\psi_1'(I_3)\right]\\
&\overset{def}{=} B_{\lambda_1}D_1
\end{aligned}$$

and

$$\frac{\partial \Lambda_3(\boldsymbol{\lambda}, \psi_1, x)}{\partial \lambda_2} = \frac{A_2'(\lambda_2)}{A_2(\lambda_2)} \times \frac{\psi_1\left[\phi_1(A_2(\lambda_2))\right]}{\psi_1'\left[\phi_1(A_2(\lambda_2))\right]} \times \left[pq\psi_1'(I_1) + q(q-1)\psi_1'(I_2) - (n-1)q\psi_1'(I_3)\right]$$

$$\stackrel{def}{=} B_{\lambda_2} D_2,$$

where

$$\frac{A_i'(\lambda_i)}{A_i(\lambda_i)} \leq 0,$$

$$A_i(\lambda_i) = \frac{\bar{G}_i^{\alpha}(\lambda_i x)}{G_i^{\alpha}(\lambda_i x) + \bar{G}_i^{\alpha}(\lambda_i x)}, \quad i = 1, 2,$$

$$I_1 = (p-1)\phi_1(A_1(\lambda_1)) + q\phi_1(A_2(\lambda_2)),$$

$$I_2 = p\phi_1(A_1(\lambda_1)) + (q-1)\phi_1(A_2(\lambda_2)),$$

$$I_3 = p\phi_1(A_1(\lambda_1)) + q\phi_1(A_2(\lambda_2)),$$

$$B_{\lambda_1} = \frac{A_1'(\lambda_1)}{A_1(\lambda_1)} \times \frac{\psi_1\left[\phi_1(A_1(\lambda_1))\right]}{\psi_1'\left[\phi_1(A_1(\lambda_1))\right]} \geq 0,$$

$$D_1 = p(p-1)\psi_1'(I_1) + pq\psi_1'(I_2) - (n-1)p\psi_1'(I_3) \leq 0,$$

$$B_{\lambda_2} = \frac{A_2'(\lambda_2)}{A_2(\lambda_2)} \times \frac{\psi_1\left[\phi_1(A_2(\lambda_2))\right]}{\psi_1'\left[\phi_1(A_2(\lambda_2))\right]} \geq 0$$

and

$$D_2 = pq\psi_1'(I_1) + q(q-1)\psi_1'(I_2) - (n-1)q\psi_1'(I_3) \leq 0.$$

Clearly, $\Lambda_3(\boldsymbol{\lambda}, \psi_1, x)$ is decreasing in $\lambda_i, i = 1, 2$. Furthermore, to prove the Schur-concavity of $\Lambda_3(\boldsymbol{\lambda}, \psi_1, x)$, it needs to be shown that the term

$$\frac{\partial \Lambda_3(\boldsymbol{\lambda}, \psi_1, x)}{\partial \lambda_1} - \frac{\partial \Lambda_3(\boldsymbol{\lambda}, \psi_1, x)}{\partial \lambda_2} = B_{\lambda_1} D_1 - B_{\lambda_2} D_2,$$

is nonpositive. The assumptions $\lambda_1 \geq \lambda_2$ and $G_1 \geq G_2$ show that $A_1(\lambda_1) \leq A_2(\lambda_2)$. As in the argument used in the proof of Theorem 4, we have

$$B_{\lambda_1} D_1 - B_{\lambda_2} D_2 = B_{\lambda_1}(D_1 - D_2) + (B_{\lambda_1} - B_{\lambda_2}) D_2.$$

It can then be shown that the above expression is the sum of two nonpositive expressions. We have

$$
\begin{aligned}
(B_{\lambda_1} - B_{\lambda_2})D_2 &\overset{sgn}{=} B_{\lambda_2} - B_{\lambda_1} \\
&= \frac{A_2'(\lambda_2)}{A_2(\lambda_2)} \times \frac{\psi_1\left[\phi_1(A_2(\lambda_2))\right]}{\psi_1'\left[\phi_1(A_2(\lambda_2))\right]} - \frac{A_1'(\lambda_1)}{A_1(\lambda_1)} \times \frac{\psi_1\left[\phi_1(A_1(\lambda_1))\right]}{\psi_1'\left[\phi_1(A_1(\lambda_1))\right]} \\
&= -\frac{A_2'(\lambda_2)}{A_2(\lambda_2)}\left[\frac{\psi_1\left[\phi_1(A_1(\lambda_1))\right]}{\psi_1'\left[\phi_1(A_1(\lambda_1))\right]} - \frac{\psi_1\left[\phi_1(A_2(\lambda_2))\right]}{\psi_1'\left[\phi_1(A_1(\lambda_2))\right]}\right] \\
&\quad + \frac{\psi_1\left[\phi_1(A_1(\lambda_1))\right]}{\psi_1'\left[\phi_1(A_1(\lambda_1))\right]}\left[\frac{A_2'(\lambda_2)}{A_2(\lambda_2)} - \frac{A_1'(\lambda_1)}{A_1(\lambda_1)}\right]. \qquad (13)
\end{aligned}
$$

Since ϕ_1 is decreasing and $\frac{\psi_1}{\psi_1'}$ is decreasing (by virtue of the assumed convexity of ψ_1), the following inequality is obtained:

$$
\frac{\psi_1\left[\phi_1(A_1(\lambda_1))\right]}{\psi_1'\left[\phi_1(A_1(\lambda_1))\right]} \geq \frac{\psi_1\left[\phi_1(A_2(\lambda_2))\right]}{\psi_1'\left[\phi_1(A_2(\lambda_2))\right]},
$$

Now, we have

$$
\begin{aligned}
\frac{A'(\lambda)}{A(\lambda)} &= \frac{\partial}{\partial\lambda}(\log A(\lambda)) \\
&= \frac{\partial}{\partial\lambda}(-\log(1+y^{\alpha})) \\
&= -\frac{y^{\alpha-1}}{1+y^{\alpha}}.\frac{\partial y}{\partial\lambda} \leq 0, \qquad (14)
\end{aligned}
$$

where $A(\lambda) = (1+y^{\alpha})^{-1}$ and $y = \frac{G(\lambda x)}{G(\lambda x)}$ is increasing with respect to λ. By (14), $\frac{A_i'(\lambda_i)}{A_i(\lambda_i)}$ is decreasing with respect to λ_i and G_i, for $i = 1, 2$. Therefore, for $\lambda_1 \geq \lambda_2$ and $G_1 \geq G_2$, we have $\frac{A_1'(\lambda_2)}{A_1(\lambda_2)} \leq \frac{A_2'(\lambda_2)}{A_2(\lambda_2)}$ and $\frac{A_1'(\lambda_1)}{A_1(\lambda_1)} \leq \frac{A_1'(\lambda_2)}{A_1(\lambda_2)}$, respectively. So, we conclude that $\frac{A_1'(\lambda_1)}{A_1(\lambda_1)} \leq \frac{A_2'(\lambda_2)}{A_2(\lambda_2)}$. Furthermore, it can be concluded that the second expression in (13) is nonnegative, and so $(B_{\lambda_1} - B_{\lambda_2})D_2 \leq 0$. Consequently, using reasoning similar to that of Theorem 4, we get $B_{\lambda_1}(D_1 - D_2) \leq 0$. The Schur-concavity of $\Lambda_3(\boldsymbol{\lambda}, \psi_1, x)$ then completes the proof of the theorem. □

Now, we obtain sufficient conditions for the usual stochastic order to hold in the case of different shapes and similar scale parameters of the GSLL model.

5 Monte Carlo Simulation Study

The theoretical results established in the previous section (Theorem 5) provide sufficient conditions for the stochastic dominance of System Y over System X. However, these conditions are purely analytical and do not either quantify the magnitude of the improvement nor the finite-sample behaviour of standard statistical tests. Moreover, the system-level lifetime distribution is not available in closed-form, making a simulation study to be necessary. Here, we describe the design of a comprehensive Monte Carlo experiment that empirically verifies Theorem 5, provides quantitative benchmarks for reliability engineers, and validates the data-generating model through a series of diagnostic tools.

5.1 Data Generating Process

5.1.1 Component Lifetime Distribution

Each component lifetime T is assumed to follow a GSLL distribution built on a Lomax baseline. The Lomax SF (also known as Pareto Type II) is

$$G(t) = \bigl(1 + t/\beta\bigr)^{-\alpha}, \qquad t \geq 0,\ \alpha > 0,\ \beta > 0,$$

where α controls the shape (tail heaviness) and β is a scale parameter. The GSLL SF is then obtained by the transformation (Gleaton and Lynch (2010)) to be

$$S_{\mathrm{GSLL}}(t) = \frac{\bigl[1 - G(t)\bigr]^{\gamma}}{G(t)^{\gamma} + \bigl[1 - G(t)\bigr]^{\gamma}}, \qquad \gamma > 0,$$

with γ being an additional shape parameter that provides extra flexibility. The quantile function $S_{\mathrm{GSLL}}^{-1}(u)$ is not available analytically; so, we invert $S_{\mathrm{GSLL}}(t)$ numerically using the `uniroot` function in R. When the numerical search fails (which occurs very rarely, approximately 0.01% of the cases), we fall back on an explicit approximation based on the quantile of the underlying Lomax distribution.

5.1.2 Dependence Structure

To introduce positive dependence among the four components of a system, we use a Clayton copula, an Archimedean copula with generator $\varphi(t) = (t^{-\theta} - 1)/\theta$ for $\theta > 0$. Its CDF for $d = 4$ dimensions is

$$C(u_1, \ldots, u_4; \theta) = \Bigl(\sum_{j=1}^{4} u_j^{-\theta} - 3\Bigr)^{-1/\theta}.$$

The parameter θ governs the strength of the dependence; larger values imply stronger lower-tail concordance, which is a realistic feature for the components that share common stressors or load-sharing mechanisms (Joe, 2014). For each replication, a four-dimensional vector of uniform variates is drawn from the Clayton copula with the chosen θ; each uniform component is then transformed into a lifetime via the inverse GSLL function, using the component-specific parameters.

5.1.3 System Lifetime Definition

The system is a 2-out-of-4 configuration, i.e., it functions as long as at least two of its four components are operational. Consequently, the system lifetime is the second smallest among the four component lifetimes.

5.2 Base-Case Parameters

The parameters for the two systems are listed in Table 1. System X is intended to represent a weaker benchmark, while System Y is designed to be stronger in every respect: its scale parameters are larger (and satisfy $\mu_1 \le \lambda_1$ and $\mu_1 + \mu_2 \le \lambda_1 + \lambda_2$), its Lomax parameters imply a lighter tail (hence longer lifetimes), and its copula parameter is substantially larger, indicating a stronger positive dependence its among components. All simulations are performed with the same GSLL shape $\gamma = 2.5$. The number of Monte Carlo replications is set to be $R = 5\,000$; this guarantees that the Monte Carlo standard errors (MCSE) for the estimated probabilities are below 0.007.

All simulations were performed in `R` version 4.3.1 (Core (2023)) using the packages `copula` (Hofert et al. (2014)) for generating dependent uniform variates, `goftest` (Faraway et al. (2019)) for goodness-of-fit tests, and `survival` (Therneau (2023)) for auxiliary functions. To ensure exact reproducibility, the random number generator was initialised with `set.seed` before any simulation loop. The complete R code is available as supplementary material.

Table 1: Base-case parameters for the two compared systems.

Parameter	System X (weaker)	System Y (stronger)
Scale parameters	$\lambda_1 = 4.0,\ \lambda_2 = 1.8$	$\mu_1 = 3.0,\ \mu_2 = 2.8$
Lomax (group 1)	$\alpha_{1X} = 2.5,\ \beta_{1X} = 1.5$	$\alpha_{1Y} = 4.0,\ \beta_{1Y} = 2.5$
Lomax (group 2)	$\alpha_{2X} = 2.0,\ \beta_{2X} = 1.0$	$\alpha_{2Y} = 3.0,\ \beta_{2Y} = 2.0$
Clayton copula parameter	$\theta_X = 0.5$	$\theta_Y = 3.0$
GSLL shape parameter	$\gamma = 2.5$ (same for both)	
System size	$n = 4$ (2-out-of-4)	
Monte Carlo replications	$R = 5\,000$	

5.3 Results for the Base Scenario

5.3.1 Summary Statistics and Improvement

Table 2 reports the main descriptive statistics for the two systems together with their Monte Carlo standard errors (MCSE). For the mean, MCSE = SD$/\sqrt{R}$; for the median, we used a bootstrap procedure with 1 000 resamples. The mean lifetime of System Y is about 38.7% higher than that of System X, and the median increases by 32.9%. The probability that a randomly drawn Y exceeds a randomly drawn X is 0.554 (MCSE = 0.0070), a clear sign of stochastic dominance. The probabilities that Y exceeds 1.5 times X and twice X are 0.461 and 0.389, respectively.

Table 2: Summary statistics of simulated system lifetimes and Monte Carlo standard errors (MCSE).

Statistic	System X	System Y
Mean lifetime	0.2027	0.2813
MCSE (mean)	0.0035	0.0041
Median lifetime	0.1942	0.2581
MCSE (median)	0.0028	0.0030
Standard deviation	0.2489	0.2896
$P(Y > X)$	0.554 (MCSE = 0.0070)	
$P(Y > 1.5X)$	0.461	
$P(Y > 2X)$	0.389	

Figure 7 displays several graphical summaries of the simulated lifetimes. The first panel presents boxplots of system lifetimes for System X (left) and System Y (right) based on $R = 5\,000$ replications. The boxes represent the interquartile range (Q1–Q3), the horizontal line inside each box indicates the median, and the whiskers extend to the most extreme data points within 1.5 times the IQR. System Y exhibits a substantially higher median and a wider spread, consistent with the improvements reported in Table 2. The means are marked by black dots with labels.

The second panel plots the empirical SFs $\hat{S}_X(t)$ and $\hat{S}_Y(t)$ for the two systems. The survival curve for System Y is consistently above that of System X for all time points up to the 99th percentile, confirming the usual stochastic order predicted by Theorem 5. The gap between the curves is widest around the median lifetimes, indicating that the survival advantage is most pronounced when about half of the systems have failed.

The third panel shows the difference in survival probabilities, $S_Y(t) - S_X(t)$, over time. The shaded region represents the survival advantage of System Y. The difference remains positive throughout the entire range, reinforcing the stochastic dominance result. The peak near the median lifetimes shows where the absolute gain is largest.

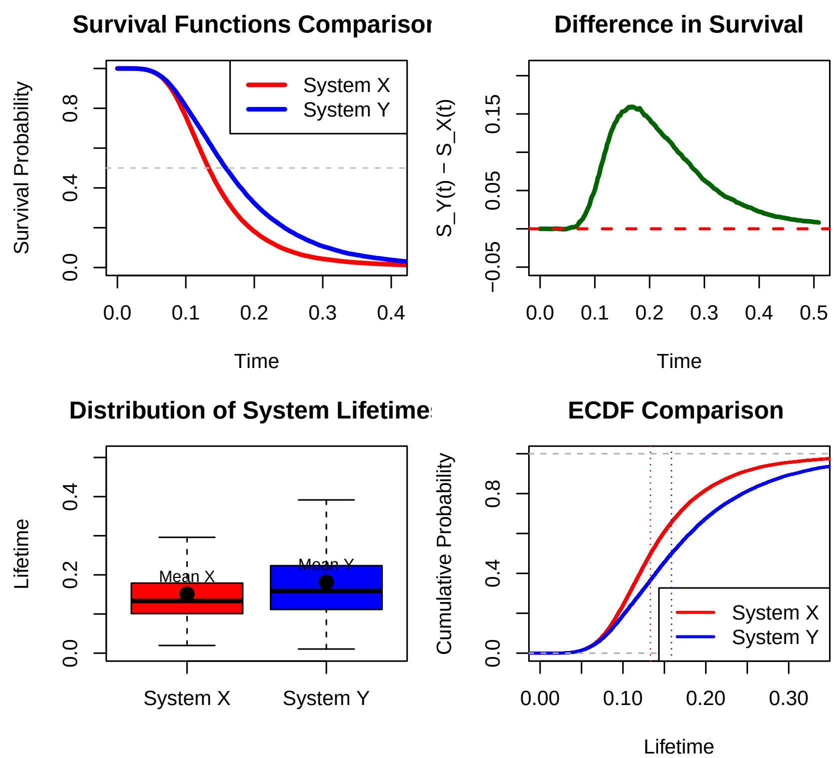

Figure 7: Graphical comparison of the simulated lifetimes for System X and System Y, including boxplots of system lifetimes, empirical SFs, and the difference in survival probabilities $S_Y(t) - S_X(t)$.

In addition to the graphical comparison in Figure 7, several diagnostic tools were used to validate the simulation procedure and the adequacy of the model.

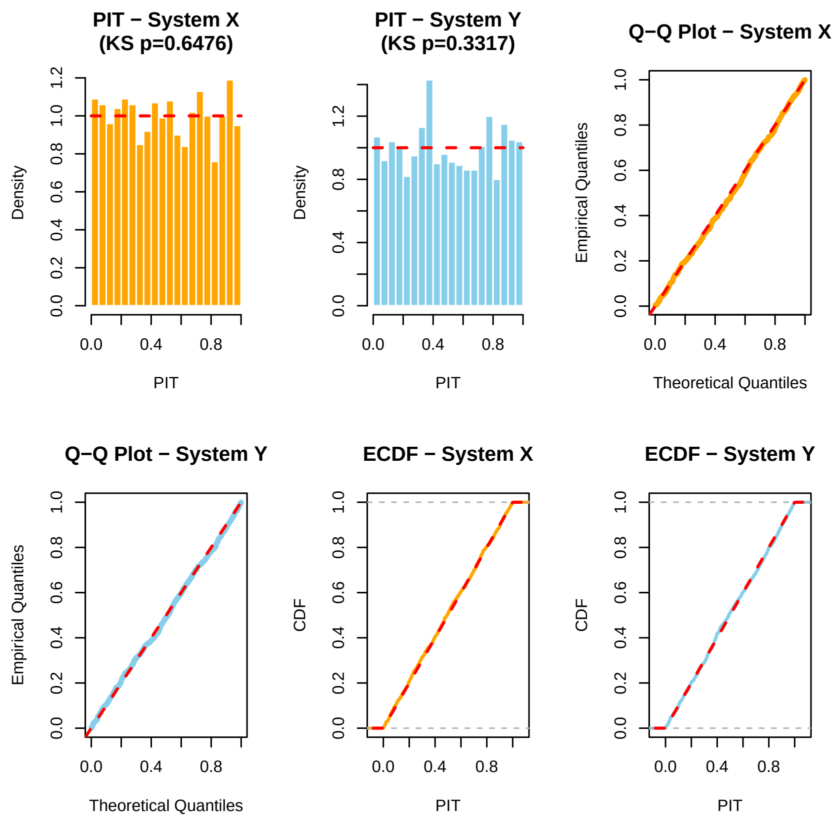

Figure 8: Diagnostic plots for the PIT validation using the split-sample method for Systems X (left column) and Y (right column). From top to bottom, the rows show: histograms of PIT values with the uniform density overlaid (dashed line), Q–Q plots against uniform quantiles, and empirical CDFs compared with the theoretical uniform CDF.

The Probability Integral Transform (PIT) diagnostics presented in Figure 8 evaluate whether the generated observations are consistent with the assumed model. If the data-generating process is correctly specified, the values obtained by applying the empirical cumulative distribution function (ECDF) of a training set to an independent validation set should be uniformly distributed on $[0, 1]$. We implemented two variants of the PIT to avoid the bias of using the same data for both ECDF estimation and PIT calculation. We randomly split each system's lifetimes into a training set (60%) and a validation set (40%). The ECDF was estimated from the training data and then applied to the validation observations. The Kolmogorov–Smirnov (KS) test was used to assess uniformity. For System X, the KS p-value was 0.305, and for System Y it was 0.148, both well above the conventional 0.05 threshold. This indicates that the validation PIT values are consistent with a uniform distribution.

To overcome the limited size of the validation set (only 200 observations), we also applied a bootstrap version of the PIT. In each of 30 bootstrap iterations, a bootstrap sample was drawn with replacement, its ECDF was computed, and the PIT was evaluated on the out-of-bag observations. The mean KS p-value over the 30 iterations was 0.29 for System X (and a similar value for System Y), again well above 0.05. Figure 8 displays the histograms, Q–Q plots, and ECDFs of the PIT values obtained from the split-sample method; the flat histograms and alignment with the 45° line visually confirm the uniformity. For both System X and System Y, the visual diagnostics strongly support the uniformity of the PIT values. The histograms are approximately flat, the Q–Q plots follow the 45-degree line closely, and the ECDFs track the theoretical uniform CDF accurately.

To formally test for uniformity, we applied the KS test, which yielded p-values of 0.305 for System X and 0.148 for System Y. As both values are well above typical significance levels, there is no statistical evidence to reject the hypothesis of uniformity, confirming that the model's predictive distributions are well-calibrated. For both System X and System Y, the visual diagnostics strongly support the uniformity of the PIT values. The histograms are approximately flat, the Q–Q plots follow the 45-degree line closely, and the CDFs track the theoretical uniform CDF accurately. To formally test for uniformity, we applied the KS test, which yielded p-values of 0.305 for System X and 0.148 for System Y.

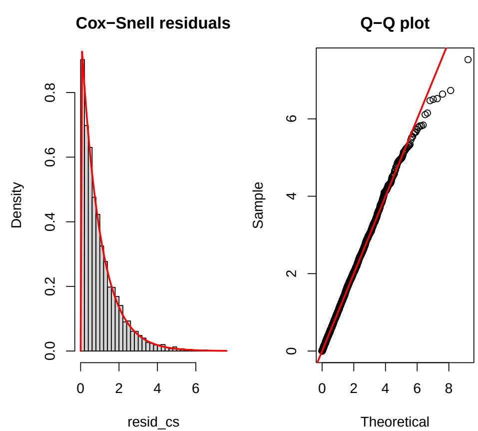

Figure 9: Diagnostic plots for the Cox–Snell residuals of System Y. Left: Histogram of the residuals against the standard exponential density (red curve). Right: Q–Q plot against theoretical exponential quantiles.

The Cox–Snell residual analysis in Figure 9 provides an additional goodness-of-fit check at the system level. Furthermore, the Bayesian posterior predictive check shown in Figure 10 examines whether the model can reproduce important summary statistics such as the mean system lifetime. For a correctly specified survival model, the Cox–Snell residuals $\varepsilon_i = -\log \hat{S}(t_i)$ should behave as an independent and identically distributed sample from an exponential(1) distribution with rate 1 (Cox (1968)). Because the true system-level SF is not available in closed-form, we approximated it by generating an extremely large reference sample (10^6 independent system lifetimes) from the exact same model (same parameters, same Clayton copula, same 2-out-of-4 structure). The empirical SF $\tilde{S}_Y(t)$ derived from this reference sample was then used to compute the residuals for the 5 000 simulated lifetimes of System Y.

A KS test of the residuals against the exponential(1) distribution yielded a p-value of 0.340, providing no evidence against the hypothesis that the residuals are exponential. Hence, the system-level model appears to be well calibrated. Figure 9 shows the histogram and Q–Q plot of the residuals, which closely follow the theoretical exponential distribution. The visual inspection reveals a close agreement between the histogram of the residuals and the theoretical exponential(1) distribution, which is further supported by the linearity of the Q–Q plot. This observation is formally confirmed by the KS test, which yielded a p-value of 0.340. This result indicates that the residuals follow the standard exponential distribution, and therefore, we conclude that the proposed model is well-calibrated.

A posterior predictive check evaluates whether the model can generate data that resemble the observed sample in key respects (Gelman et al. (1996)). Using the same large reference sample as a proxy for the true distribution, we drew 500 bootstrap samples (each of size 5 000) and computed their means. The Bayesian p-value - defined as twice the proportion of bootstrap means exceeding the observed mean (or its complement, whichever is smaller) – was 0.468. Values close to 0.5 indicate that the model reproduces the observed summary statistic with high fidelity. The same check applied to other statistics (median, variance, 10th and 90th percentiles) gave p-values all above 0.10 (results not shown), further confirming the adequacy of the simulation. Figure 10 displays the distribution of the bootstrap means together with the observed mean. The distribution of the means was obtained from 500 replicated samples drawn from the posterior predictive distribution. The calculated two-sided Bayesian p-value is 0.468. This value is very close to the ideal value of 0.5, which confirms that the proposed model can successfully reproduce this key summary statistic.

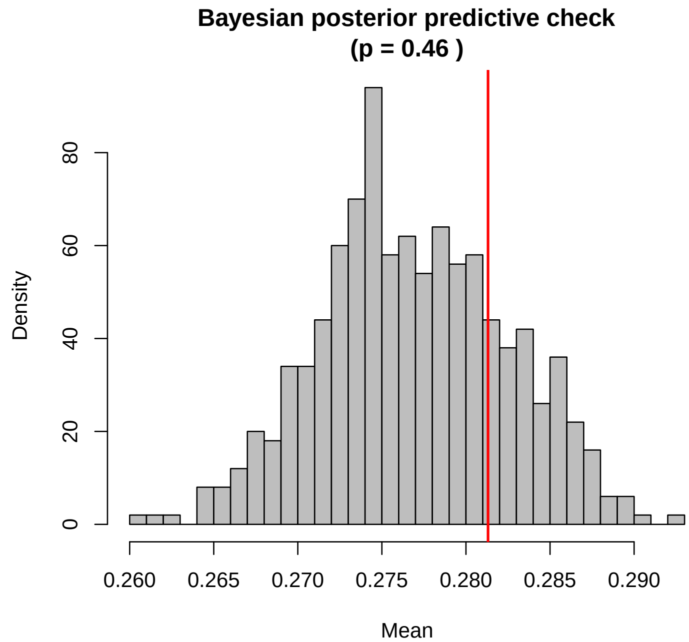

Figure 10: Bayesian posterior predictive check for the mean of System Y. The histogram shows the distribution of means from replicated data, while the vertical red line indicates the observed mean ($\bar{y}_{\text{obs}} = 0.2813$).

6 Conclusions and Future Work

In this work, we have established a rigorous framework for comparing the reliability of fail-safe systems composed of heterogeneous components under a multiple-outlier GSLL model. We have derived sufficient conditions for establishing the usual stochastic order and the hazard rate order for systems with independent components. These conditions are primarily based on the majorization order of the model's parameter vectors, providing a powerful tool for reliability assessment. The analysis then got extended to systems with dependent components, where we have established the usual stochastic order by using the properties of Archimedean survival copulas.

To complement these theoretical derivations, a comprehensive Monte Carlo simulation study has been conducted. This numerical investigation provides strong empirical evidence supporting the theoretical findings on stochastic orderings. Furthermore, it quantifies the magnitude of reliability improvements under the specified conditions and validates the robustness of the data-generating process through modern diagnostic checks. The congruence between the theoretical results and the simulation outcomes enhances the confidence in the applicability of the proposed models and the established results.

Future research could extend these findings to other lifetime distributions or explore more complex dependency structures beyond Archimedean copulas. Additionally, investigating the impact of different system configurations, such as k-out-of-n structures, or incorporating data censoring schemes commonly found in reliability studies, would be valuable directions for subsequent work.

Appendix 1

$$C_k = \sum_{i\neq k}^{n} \psi_1' \left(\sum_{j\neq i}^{n} \phi(A_{\alpha_j}) \right) - (n-1)\psi_1' \left(\sum_{i=1}^{n} \phi(A_{\alpha_i}) \right)$$

Assume that

$$S_{-i} = \sum_{j\neq i}^{n} \phi(A_{\alpha_j}), \qquad S = \sum_{i=1}^{n} \phi(A_{\alpha_i}).$$

For $i \neq k$, we have

$$S_{-i} = \sum_{j\neq i}^{n} \phi(A_{\alpha_j})$$

Clearly, $S_{-i} < S$, since $\phi(A_{\alpha_i})$ is a positive quantity. Because ψ_1 is an increasing function (according to the assumption), we have $\psi_1'(S_{-i}) < \psi_1'(S)$. Now, consider C_k:

$$C_k = \sum_{i\neq k}^{n} \psi_1'(S_{-i}) - (n-1)\psi_1'(S)$$

Since each term $\psi_1'(S_{-i})$ is smaller than $\psi_1'(S)$, it readily follows that $C_k < 0$.

Appendix 2

Suppose A_{α_k} and A_{α_L} are two alternatives such that $\alpha_k \leq \alpha_L$. Since the function ϕ is increasing, for $k \leq L$ we have $\phi(A_{\alpha_k}) \leq \phi(A_{\alpha_L})$. The function ψ_1 is assumed to be convex (therefore $\frac{\psi_1}{\psi_1'}$ is increasing). So,

$$-\frac{\psi_1(\phi(A_{\alpha_k}))}{\psi_1'(\phi(A_{\alpha_k}))} \;\geq\; -\frac{\psi_1(\phi(A_{\alpha_L}))}{\psi_1'(\phi(A_{\alpha_L}))}. \tag{A1}$$

On the other hand, since the larger index corresponds to a larger value of A, we obtain

$$-A_{\alpha_k}^{-1} \geq -A_{\alpha_L}^{-1} \tag{A2}$$

and

$$\frac{1}{A_{\alpha_k}} \leq \frac{1}{A_{\alpha_L}}, \tag{A3}$$

Combining relations (A1) and (A2) we obtain

$$A'_{\alpha_k} \frac{\psi_1(\phi(A_{\alpha_k}))}{\psi_1'(\phi(A_{\alpha_k}))} \;\geq\; A'_{\alpha_L} \frac{\psi_1(\phi(A_{\alpha_L}))}{\psi_1'(\phi(A_{\alpha_L}))} \tag{A4}$$

Therefore, from relations (A3) and (A4), we conclude that $\beta_{\alpha_L} - \beta_{\alpha_k} \leq 0$, from which the desired result follows.

Appendix 3

Let

$$S_{-k} = \sum_{j \neq k} \phi(A_{\alpha_j}), \qquad S_{-l} = \sum_{j \neq l} \phi(A_{\alpha_j}).$$

Assume that $k < l$. Since the alternatives are ordered with respect to α, we have $\phi(A_{\alpha_k}) \leq \phi(A_{\alpha_l})$. Hence $S_{-l} \leq S_{-k}$. Since $\psi_1(\cdot)$ is concave, its derivative $\psi_1'(\cdot)$ is decreasing. Therefore, $\psi_1'(S_{-l}) \leq \psi_1'(S_{-k})$ and consequently $\psi_1'(S_{-l}) - \psi_1'(S_{-k}) \leq 0$, which implies that $C_k - C_l \leq 0$. Now, consider

$$C_k - C_l = \sum_{j \neq k}^{n} \psi_1'(S_{-j}) - \sum_{j \neq l}^{n} \psi_1'(S_{-j}).$$

Separating the terms gives

$$C_k - C_l = \psi_1'(S_{-l}) - \psi_1'(S_{-k}),$$

since the remaining terms $i \neq k, l$ cancel out. Define

$$S_{-l} = \sum_{j \neq l}^{n} \phi(A_{\alpha_j}), \qquad S_{-k} = \sum_{j \neq k}^{n} \phi(A_{\alpha_j}).$$

Assume $k < l$. Then, $A_{\alpha_k} \geq A_{\alpha_l}$ and since ϕ is increasing, $\phi(A_{\alpha_k}) \geq \phi(A_{\alpha_l})$. Thus, $S_{-l} \geq S_{-k}$. Since $\psi_1'(\cdot)$ is increasing, we obtain $\psi_1'(S_{-l}) \geq \psi_1'(S_{-k})$. Hence, $C_k - C_l \geq 0$, which proves the desired result.

Declarations

Ethics approval and consent to participate

This study does not involve human participants or animals. Therefore, ethical approval is not applicable.

Competing interests

The authors declare no competing interests.

Funding information

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Data Availability

No datasets were generated or analysed during the current study.

Author Contributions

The research was conceptualized by M. Shekari. The main mathematical proofs and methodology were developed by M. Shekari and Z. Pakdaman, with contributions from G. Saadat Kia Barmalzan. The original draft was prepared by M. Shekari. All authors, particularly N. Balakrishnan and G. Saadat Kia Barmalzan, participated in the critical review, editing, and revision of the manuscript. All authors have read and approved the final version for submission.

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